Sorting monoids on Coxeter groups

A computer exploration with Sage-Combinat

Florent Hivert[1] Anne Schilling[2] Nicolas M. Thiéry[2,3]

[1]LITIS/LIFAR, Université Rouen, France

[2]University of California at Davis, USA

[3]Laboratoire de Mathématiques d'Orsay, Université Paris Sud, France

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arXiv:0711.1561v1 [math.RT]
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+ research in progress

Bubble (anti) sort algorithm

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1423

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4123

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Underlying combinatorics: right permutohedron

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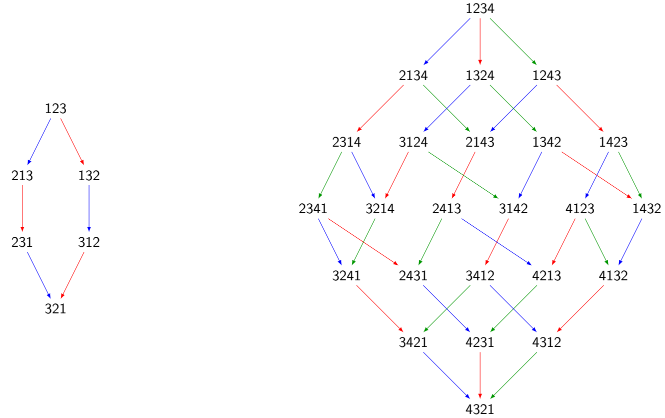

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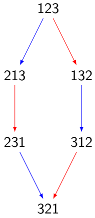

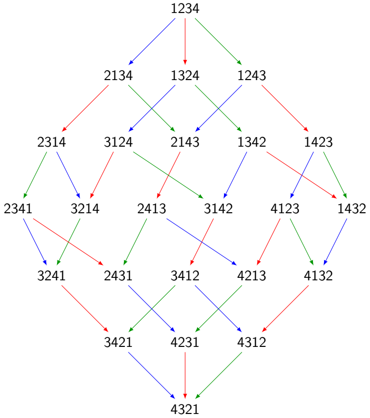

Elementary transpositions: $s_1, s_2, s_3, \ldots$

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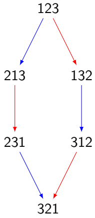

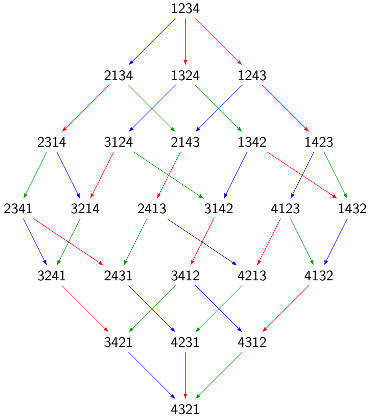

Elementary transpositions: $s_1, s_2, s_3, \ldots$

Relations: $s_i^2, (s_1 s_2)^3 = 1, (s_2 s_3)^3 = 1, (s_1 s_3)^2 = 1$

Coxeter groups

Definition (Coxeter group W)

Generators : $s_1, s_2, \dots$ (simple reflections)
Relations: $s_i^2 = 1$ and $\underbrace{s_i s_j \cdots}_{m_{i,j}} = \underbrace{s_j s_i \cdots}_{m_{i,j}}$, for $i \neq j$

- Reduced word
- Length

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Orders on words and on Coxeter group elements

Definition (Orders on words)

Let $u = u_1 \cdots u_k$ and $v = v_1 \cdots v_l$:

- u **left factor** of v if $v = u_1 \cdots u_k \cdots$
- u **right factor** of v if $v = \cdots u_1 \cdots u_k$
- u **factor** of v if $v = \cdots u_1 \cdots u_k \cdots$
- u **subword** of v if $v = \cdots u_1 \cdots u_2 \cdots u_k \cdots$

Definition (Orders on Coxeter group elements)

- Right weak order
- Left weak order
- Left-right weak order
- Bruhat order

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Blocks of permutations

Definition (Block of a permutation w)

- Type A: sub-permutation matrix
- Type free: J, K such that ${}^J w = w^K$

- Example: $w := 36475812$
- Simple permutation: cf. [Albert, Atkinson 05]
- $\{\texttt{blocks of } w\}$: sub-lattice of the Boolean lattice

Definition (HST09: Cutting poset $(W, \sqsubset)$)

$u \sqsubset w$ if $u = w^J$ with J block

- (almost) lattice
- Möbius function: inclusion-exclusion along minimal blocks

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Hecke monoid

Definition (0-Hecke monoid $H_0(W)$ of a Coxeter group W)

Generators : $\langle \pi_1, \pi_2, \dots \rangle$ (simple reflections)
Relations: $\pi_i^2 = \pi_i$ and braid relations

Theorem

$|H_0(W)| = |W|$
+ lots of nice properties

Motivation: simple combinatorial model (bubble sort)
appears in Iwahori-Hecke algebras, Schur symmetric functions, Schubert, Kazhdan-Lusztig polynomials, and Macdonald, (affine) Stanley symmetric functions, mathematical physics, Schur-Weyl duality for quantum groups, representations of $GL(\mathbb{F}_q)$, ...

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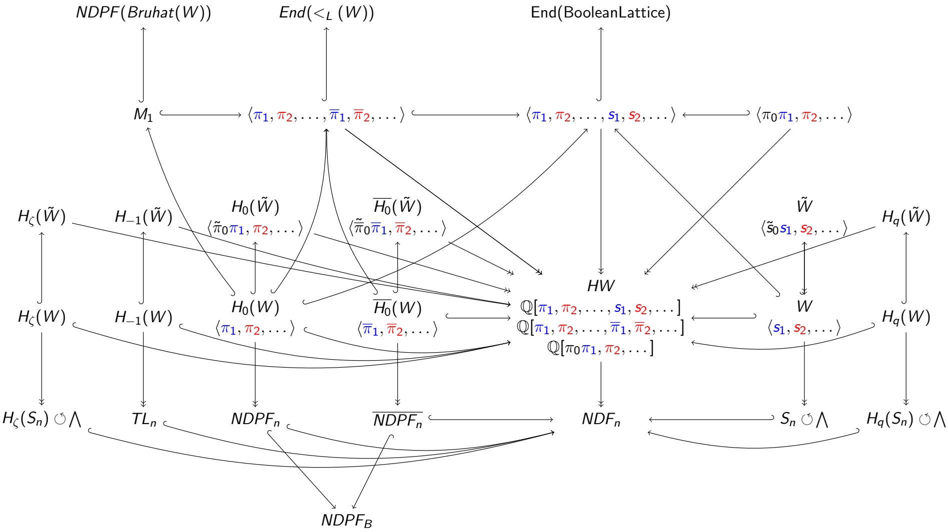
The Big Picture



The bi-Hecke monoid

Question

Size of $M(W) = \langle \pi_1, \pi_2, \ldots, \overline{\pi}_1, \overline{\pi}_2, \ldots \rangle$

$|M(S_n)| = 1, 3, 23, 477, 31103, ?$

- How to attack such a problem?
- Generators and relations?
- Representation theory?

Theorem (HST08)

$M(W)$ admits $|W|$ simple / indecomposable projective modules

- Why do we care?

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Theorem (HST08)

$M(W)$ *admits* $|W|$ *simple / indecomposable projective modules*

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Representation theory of algebras

Module: vector space V with a morphism $M \mapsto \mathrm{End}(V)$
Simple module: V contains no nontrivial submodule
Indecomposable module: V cannot be written as $V = V_1 \oplus V_2$
Projective module: $V \oplus \cdots = \mathbb{C}[M] \oplus \cdots \oplus \mathbb{C}[M]$

Theorem (See e.g. Curtis-Reiner)

Simple modules $\leftrightarrow$ indecomposable projective modules
Dimension formula, ...

Key role of idempotents:

- eV projective module
- If $f = uev$ then fM is isomorphic to a submodule of eM

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Definition (J-(pre)order)

$x \leq_J y$ iff $x = uyv$, for some $u, v \in M$
$x, y \in M$ are in the same J-class if $x \leq_J y$ and $y \leq_J x$
A J-class is regular iff it contains an idempotent

Theorem (See e.g. Ganyushkin, Mazorchuk, Steinberg 07)

The regular J-classes (essentially) determine the simple modules.

$\Longrightarrow$ Combinatorial Representation Theory

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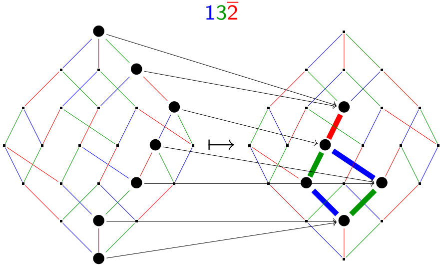


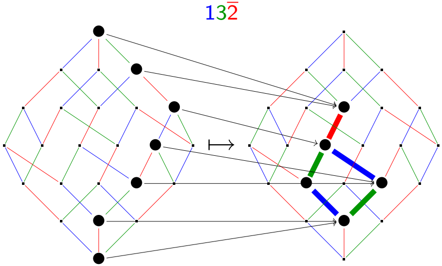
Lemma

For $f \in M(W)$ and $w \in W$: $\quad (s_i w).f = w.f \quad$ *or* $\quad s_i\,(w.f)$

Proof.

Exchange property □

Key combinatorial lemma



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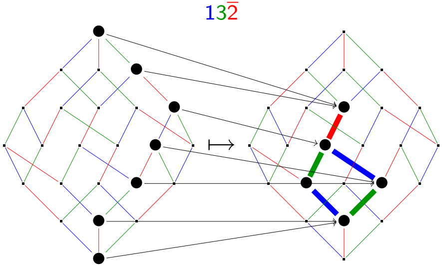

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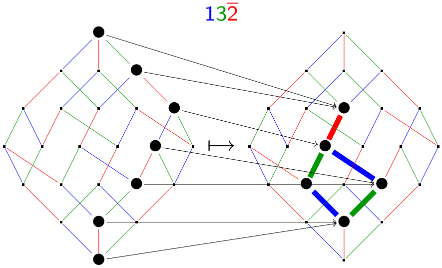

Corollary

- *If $w = uv$, then $(uv).f = u' \, (v.f)$, where $u' <_B u$*
- *Preservation of left order: $u \leq_L v \Longrightarrow u.f \leq_L v.f$*
- *Preservation of Bruhat order: $u \leq_B v \Longrightarrow u.f \leq_B v.f$*
- *f in $M(W)$ is determined by its fibers and $f(1)$*

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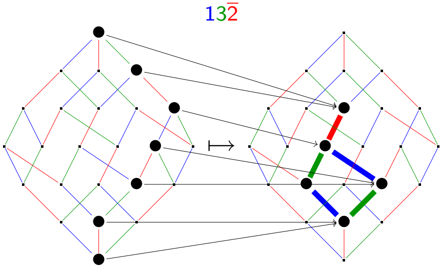

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- *If $w = uv$, then $(uv).f = u' \, (v.f)$, where $u' <_B u$*
- *Preservation of left order: $u \leq_L v \Longrightarrow u.f \leq_L v.f$*
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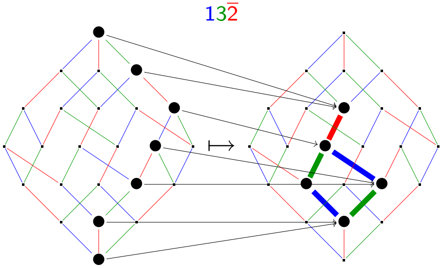
- *f in $M(W)$ is determined by its fibers and $f(1)$*

Key combinatorial lemma



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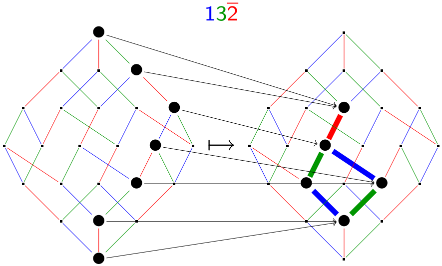

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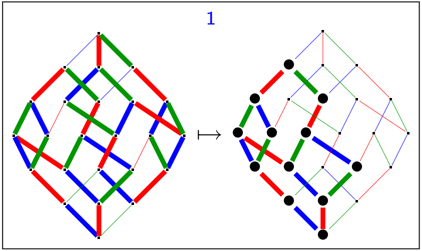

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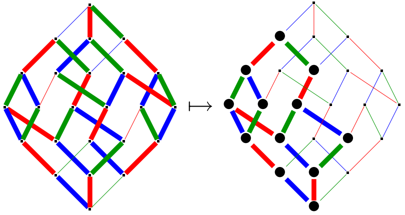

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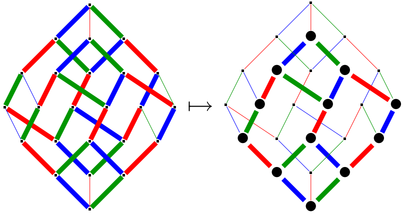

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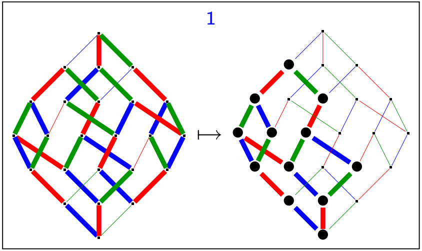

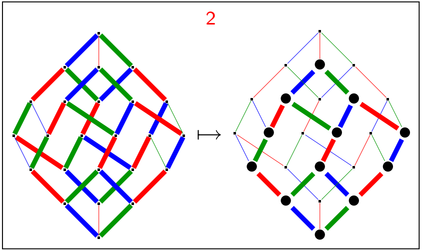

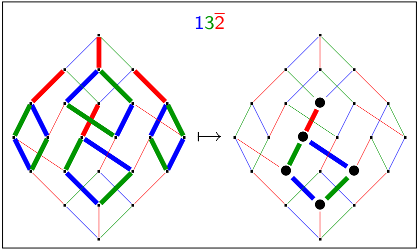

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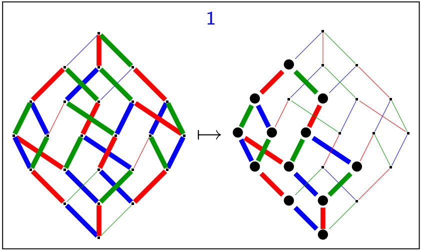

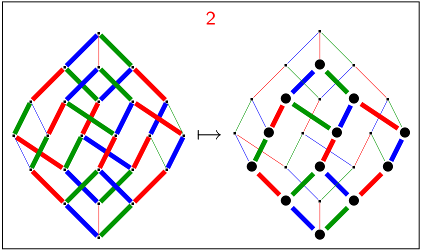

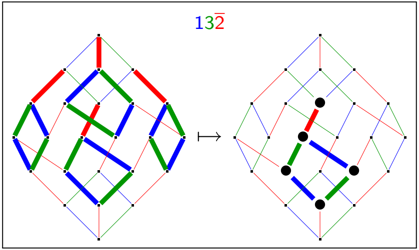

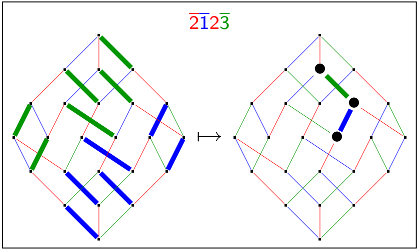

Representation theory of $M(W)$

Theorem (HST'08)

$M(W)$ admits $|W|$ simple modules

Sketch of proof.

- M acts transitively on intervals $[u, v]_L$
- The image set of an idempotent is an interval $[u, v]_L$
- $\exists!$ e_w idempotent with image set $[1, w]_L$, for any $w \in W$
- $(e_w)_{w \in W}$: transversal of the regular J-classes
  - $f = uev$ if and only if $\mathrm{im}(f)$ is a subinterval of $\mathrm{im}(e)$
- The groups $e_w M e_w$ are trivial

□

Problem

Dimension of simple and projective modules?

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Dimension of simple and projective modules?

The “Borel” submonoid M_1

Definition

Submonoid $M_1 := \{f \in M, f(1) = 1\}$

Properties (HST'09)

- *Generated by e_w for w grassmanian*
- *Weakly increasing and contracting on Bruhat $\Longrightarrow$ J-trivial*
- *Idempotents: $(e_w)_{w \in W}$*
- *$|W|$ simple modules of dimension 1*
- *Semi simple quotient: monoid algebra of $(W, \vee_L)$*
- *Conjugacy order among idempotents: $<_{LR}$*
- $\dim P_w = |\{f \in M_1, f(w) = w \dots\}|$?

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Inducing those results to M?

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Representation theory of J-trivial monoids

Theorem (HST'09)

Combinatorial description of:

- *Simple modules*
- *Projective modules*
- *Cartan matrix*
- *Quiver*
- *q-Cartan matrix (in progress)*

in term of some statistic on M

Question

Induction from "Borel" submonoids?

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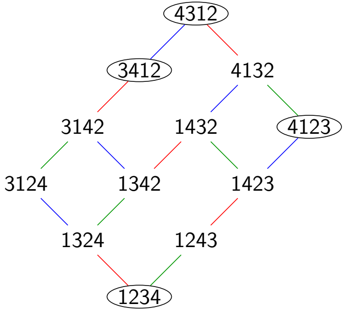
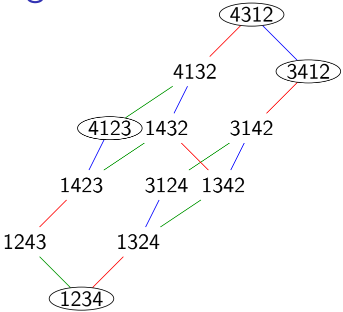
Translation algebras





Definition (Translation algebra)

$T_w := \mathbb{Q}[\pi_1, \pi_2, \ldots, \overline{\pi}_1, \overline{\pi}_2, \ldots]$ acting on $\mathbb{Q}.[1, w]_R$

- Blocks: $J = \{\}, \{1, 2\}, \{3\}, \{1, 2, 3\}$ $\Longrightarrow$ Submodules P_J
- T_w: max. algebra stabilizing all P_J $\Longrightarrow$ Repr. theory
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- Dimension: inclusion-exclusion along the cutting poset
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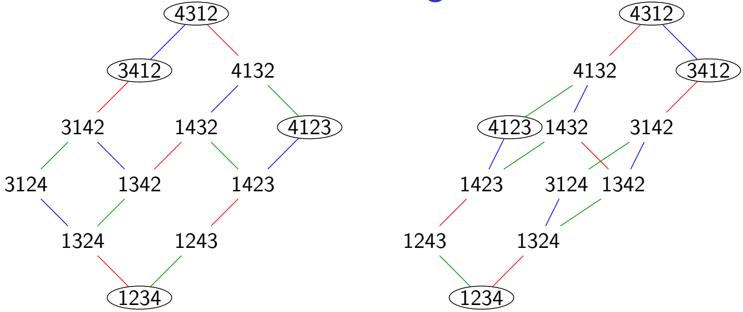

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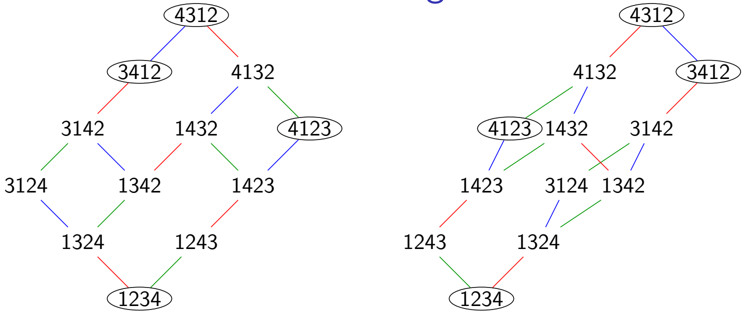

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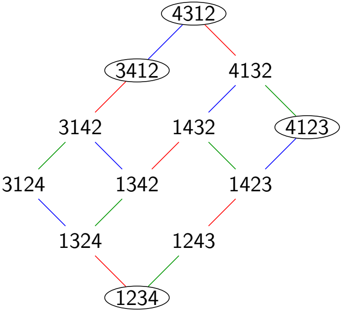

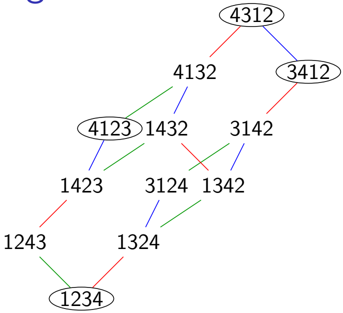

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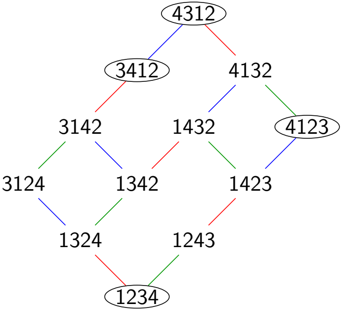

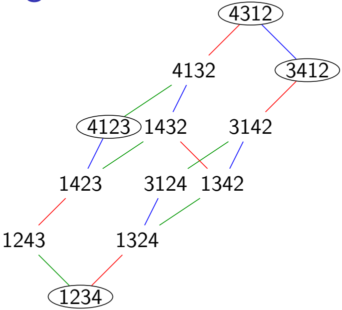

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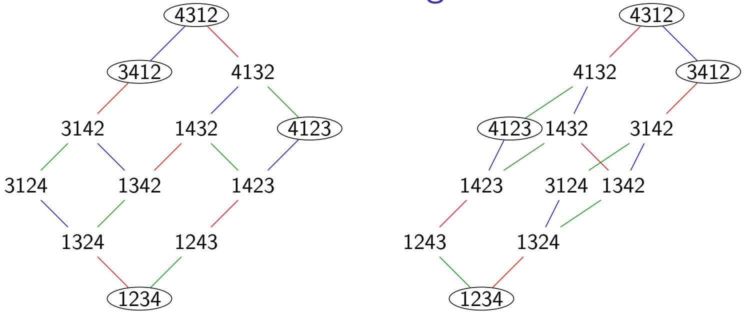

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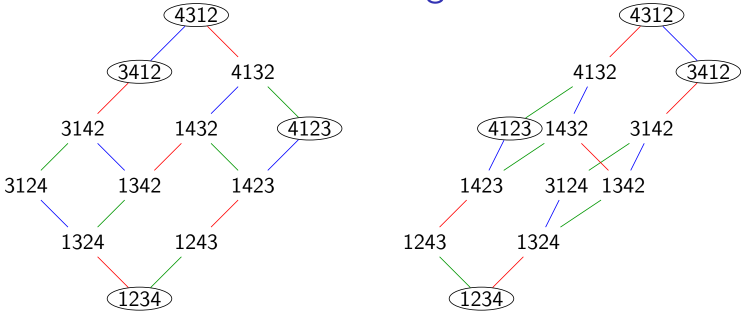

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General strategy:

- Find combinatorial models for algebras and representations
- As simple as possible, but no simpler
- Concrete and effective
- Use representation theory and computer exploration as a guide
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