

# Sorting monoids on Coxeter groups

## A computer exploration with Sage-Combinat

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arXiv:0711.1561v1 [math.RT]

arXiv:0804.3781v1 [math.RT]

arXiv:0912.2212v1 [math.CO]

+ research in progress

## Sage-Combinat ([combinat.sagemath.org](http://combinat.sagemath.org))

- 50+ research articles
- NSF Sponsored
- Sage: 300 tickets / 100k lines integrated in Sage
- MuPAD: 115k lines of MuPAD, 15k lines of C++, 32k lines of tests, 600 pages of doc
- Nicolas Borie, Daniel Bump, Jason Bandlow, Adrien Boussicault, Vincent Delecroix, Paul-Olivier Dehaye, Tom Denton, Dan Drake, Teresa Gomez Diaz, Mike Hansen, Ralf Hemmecke, Florent Hivert, Brant Jones, Sébastien Labbé, Yann Laigle-Chapuy, Andrew Mathas, Gregg Musiker, Steven Pon, Franco Saliola, Anne Schilling, Mark Shimozono, Nicolas M. Thiéry, Justin Walker, Qiang Wang, Mike Zabrocki, ...

# Bubble (anti) sort algorithm

1234

1234

1243

1423

4123

4123

4132

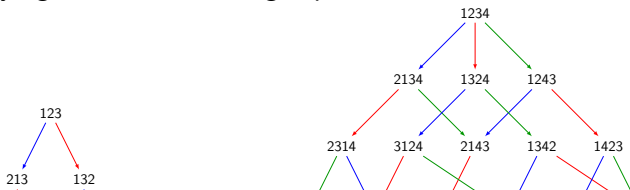
4312

4312

4321

4321

Underlying combinatorics: right permutohedron



# Coxeter groups

## Definition (Coxeter group $W$ )

Generators :  $s_1, s_2, \dots$  (simple reflections)

Relations:  $s_i^2 = 1$     and     $\underbrace{s_i s_j \cdots}_{m_{i,j}} = \underbrace{s_j s_i \cdots}_{m_{i,j}},$  for  $i \neq j$

- Reduced word
- Length

## Orders on words and on Coxeter group elements

### Definition (Orders on words)

Let  $u = u_1 \cdots u_k$  and  $v = v_1 \cdots v_l$ :

- $u$  **left factor** of  $v$  if  $v = u_1 \cdots u_k \cdots$
- $u$  **right factor** of  $v$  if  $v = \cdots u_1 \cdots u_k$
- $u$  **factor** of  $v$  if  $v = \cdots u_1 \cdots u_k \cdots$
- $u$  **subword** of  $v$  if  $v = \cdots u_1 \cdots u_2 \cdots u_k \cdots$

### Definition (Orders on Coxeter group elements)

- Right weak order
- Left weak order
- Left-right weak order
- Bruhat order

## Blocks of permutations

### Definition (Block of a permutation $w$ )

- Type  $A$ : sub-permutation matrix
- Type free:  $J, K$  such that  $W_J w = w W_K$
- Example:  $w := 36475812$
- Simple permutation: cf. [Albert, Atkinson 05]
- $\{\text{blocks of } w\}$ : sub-lattice of the Boolean lattice

### Definition (HST09: Cutting poset $(W, \sqsubset)$ )

$u \sqsubset w$  if  $u = w^J$  with  $J$  block

- (almost) lattice
- Möbius function: inclusion-exclusion along minimal blocks

# Hecke monoid

Definition (0-Hecke monoid  $H_0(W)$  of a Coxeter group  $W$ )

Generators :  $\langle \pi_1, \pi_2, \dots \rangle$  (simple reflections)

Relations:  $\pi_i^2 = \pi_i$    and braid relations

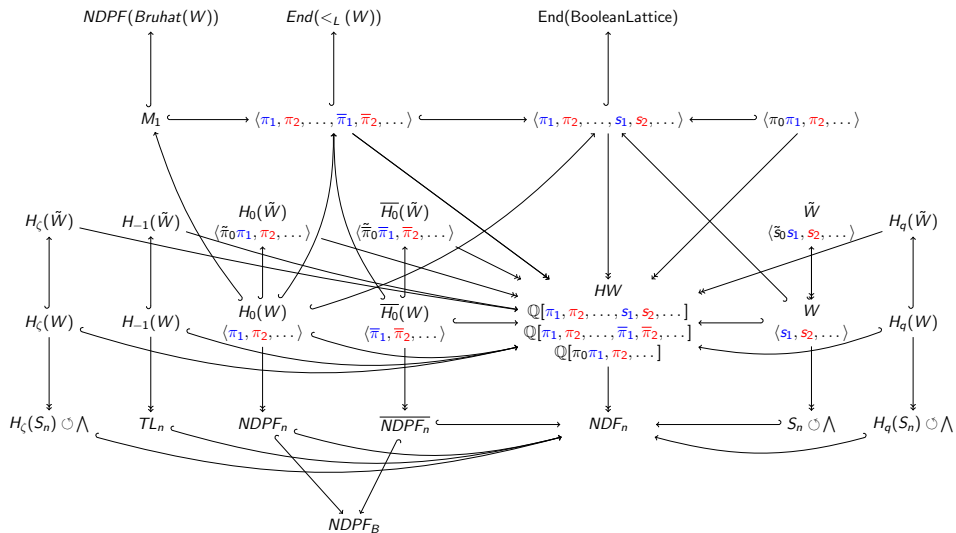
Theorem

$|H_0(W)| = |W|$

+ *lots of nice properties*

**Motivation:** simple combinatorial model (bubble sort)  
 appears in Iwahori-Hecke algebras, Schur symmetric functions,  
 Schubert, Kazhdan-Lusztig polynomials, and Macdonald, (affine)  
 Stanley symmetric functions, mathematical physics, Schur-Weyl  
 duality for quantum groups, representations of  $GL(\mathbb{F}_q)$ , ...

# The Big Picture



## The bi-Hecke monoid

### Question

Size of  $M(W) = \langle \pi_1, \pi_2, \dots, \bar{\pi}_1, \bar{\pi}_2, \dots \rangle$

$|M(S_n)| = 1, 3, 23, 477, 31103, ?$

- How to attack such a problem?
- Generators and relations?
- Representation theory?

### Theorem (HST08)

$M(W)$  admits  $|W|$  simple / indecomposable projective modules

- Why do we care?

$$|M(W)| = \sum_{w \in W} \dim S_w \cdot \dim P_w$$

## Representation theory of algebras

**Module:** vector space  $V$  with a morphism  $M \mapsto \text{End}(V)$

**Simple module:**  $V$  contains no nontrivial submodule

**Indecomposable module:**  $V$  cannot be written as  $V = V_1 \oplus V_2$

**Projective module:**  $V \oplus \dots = \mathbb{C}[M] \oplus \dots \oplus \mathbb{C}[M]$

Theorem (See e.g. Curtis-Reiner)

*Simple modules*  $\leftrightarrow$  *indecomposable projective modules*

*Dimension formula, ...*

Key role of idempotents:

- $eV$  projective module
- If  $f = uev$  then  $fM$  is isomorphic to a submodule of  $eM$

## Representation theory of monoids

### Definition ( $J$ -(pre)order)

$x \leq_J y$  iff  $x = uyv$ , for some  $u, v \in M$

$x, y \in M$  are in the same  $J$ -class if  $x \leq_J y$  and  $y \leq_J x$

A  $J$ -class is regular iff it contains an idempotent

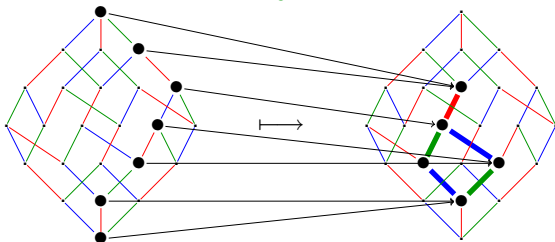
Theorem (See e.g. Ganyushkin, Mazorchuk, Steinberg 07)

*The regular  $J$ -classes (essentially) determine the simple modules.*

$\implies$  Combinatorial Representation Theory

# Key combinatorial lemma

## Key combinatorial lemma

13 $\bar{2}$ 

## Lemma

For  $f \in M(W)$  and  $w \in W$ :  $(s_i w).f = w.f$  or  $s_i(w.f)$

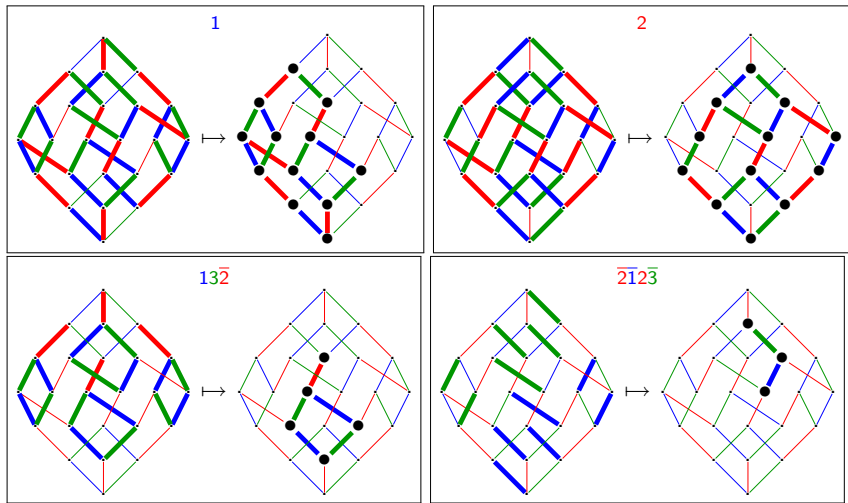
## Proof.

Exchange property / associativity □

## Corollary

- If  $w = uv$ , then  $(uv).f = u'(v.f)$ , where  $u' <_B u$
- Preservation of left order:  $u <_B v \implies u.f <_B v.f$

## Some elements of the monoid



## Representation theory of $M(W)$

### Theorem (HST'08)

$M(W)$  admits  $|W|$  simple modules

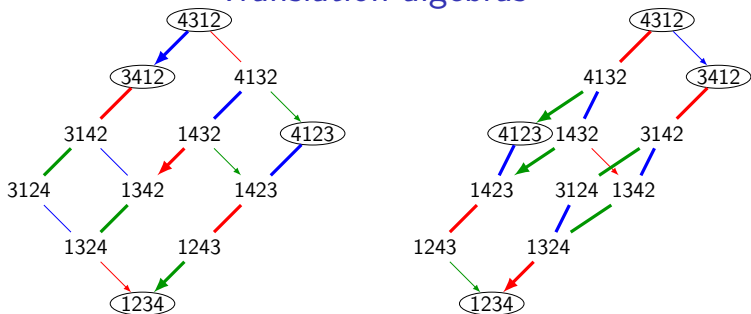
### Sketch of proof.

- $M$  acts transitively on intervals  $[u, v]_L$
- The image set of an idempotent is an interval  $[u, v]_L$
- $\exists!$   $e_w$  idempotent with image set  $[1, w]_L$ , for any  $w \in W$
- $(e_w)_{w \in W}$ : transversal of the regular  $J$ -classes
  - $f = uev$  if and only if  $\text{im}(f)$  is a subinterval of  $\text{im}(e)$
- The groups  $e_w M e_w$  are trivial □

### Problem

*Dimension of simple and projective modules?*

## Translation algebras



## Definition (Translation algebra)

$$T_w := \mathbb{Q}[\pi_1, \pi_2, \dots, \bar{\pi}_1, \bar{\pi}_2, \dots] \text{ acting on } \mathbb{Q}\langle 1, w \rangle_R$$

- Blocks:  $J = \{\}, \{1, 2\}, \{3\}, \{1, 2, 3\} \implies$  Submodules  $P_J$
- $T_w$ : max. algebra stabilizing all  $P_J \implies$  Repr. theory
- $T_w$  quotient of  $\mathbb{Q}[M(W)]$ ; top: simple module  $S_w$  of  $M$
- Dimension: inclusion-exclusion along the cutting poset
- Generating series calculation?

## The “Borel” submonoid $M_1$

### Definition

Submonoid  $M_1 := \{f \in M, f(1) = 1\}$

### Properties (HST'09)

- *Generated by  $e_w$  for  $w$  grassmanian*
- *Weakly increasing and contracting on Bruhat  $\implies J$ -trivial*
- *Idempotents:  $(e_w)_{w \in W}$*
- *$|W|$  simple modules of dimension 1*
- *Semi simple quotient: monoid algebra of  $(W, \vee_L)$*
- *Conjugacy order among idempotents:  $<_L$*
- *$\dim P_w = |\{f \in M_1, f(w) = w\dots\}|$  ?*

### Problem

*Inducing these results to  $M$ ?*

# Representation theory of J-trivial monoids

## Theorem (HST'09)

*Combinatorial description of:*

- *Simple modules*
- *Projective modules*
- *Cartan matrix*
- *Quiver*
- *q-Cartan matrix (in progress)*

*in term of some statistic on  $M$*

## Question

*Induction from “Borel” submonoids?*

# The path algebra of a Quiver

## Definition

- Quiver: (edge labeled) graph  $Q = (V, E)$
- path of length  $l$  (possibly  $= 0$ )

$$p := (v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \cdots \xrightarrow{e_l} v_l)$$

such that  $e_i$  is an edge from  $v_{i-1}$  to  $v_i$ .

- path algebra (category): product = concatenation if last and first vertex matches else 0.

TODO : Add an example

# Structure theorem for finite dimensional algebras

## Definition

**admissible ideal**: included in the ideal of path of length  $\geq 2$ .

## Theorem

*For any (elementary) algebra  $A$ , there is a **unique quiver**  $Q$  such that  $A$  is the quotient of  $\mathbb{C}Q$  by an admissible ideal  $I$ .*

Elementary algebras: simple module are all 1-dimensional.

Note: first order approximation of the algebra.

Note: the ideal  $I$  is far from being unique.

## Vertices of the Quiver ?

Decomposition of the identity:

$$1 = \sum_e f_e \quad \text{and} \quad f_e f_{e'} = \delta_{ee'} f_e;$$

Theorem (HST 09)

*Association:  $e \in M \mapsto f_e \in \mathbb{C}[M]$ , such that  
Moreover  $f_e = e + \text{smaller terms} \dots$*

Theorem

*The vertices of the Quiver are naturally indexed by the idempotents of the monoid.*

## Cartan's invariants

Matrix decomposition of the algebra  $x \in \mathbb{C}[M]$ :

$$x = \sum_{e_1, e_2} x_{e_1, e_2} \quad \text{where} \quad x_{e_1, e_2} = f_{e_1} x f_{e_2} .$$

$$(xy)_{e_1, e_2} = \sum_e x_{e_1, e} y_{e, e_2}$$

# Automorphism sub-monoids and factorizations

## Definition (Automorphism sub-monoids)

$$\text{rAut}(x) := \{u \in M \mid xu = x\}$$

## Proposition

*There exists a unique idempotent  $\text{rfix}(x)$  such that*

$$\text{rAut}(x) = \{u \in M \mid \text{rfix}(x) \leq_J u\}.$$

Same one the left ( $\text{lAut}(x), \text{lfix}(x)$ ).

## Corollary: Cartan invariants + vertex of the quiver

### Theorem

*Cartan's invariants:*

$$\dim(f_{e_1} \mathbb{C}[M] f_{e_2}) = \#\{x \in M \mid \text{lfix}(x) = e_1 \text{ and } \text{rfix}(x) = e_2\}.$$

# Factorizations

## Definition

Let  $x \in M$  non idempotent and  $e := \text{and}$  and  $f := \text{rfix}(x)$ .

A factorization  $x = uv$  is

- **non-trivial** if  $eu \neq e$  and  $vf \neq f$   
equivalently if  $u \notin \text{lAut}(x)$  and  $v \notin \text{rAut}(x)$ ;
- **compatible** if  $u$  and  $v$  are non-idempotent and

$$\text{lfix}(u) = e, \quad \text{rfix}(v) = f \quad \text{and} \quad \text{rfix}(u) = \text{lfix}(v);$$

## Factorizations and irreducible

### Proposition

- *compatible*  $\Rightarrow$  *non-trivial*;
- *non-trivial and  $j$ -minimal*  $\Rightarrow$  *compatible*.

### Definition

$x \in M$  non idempotent is **irreducible** if there is no non-trivial factorizations  $x = uv$ .

# The Quiver of (the algebra of) a $j$ -trivial monoid

## Theorem

*The quiver of the algebra of  $M$  is the following:*

- *There is one vertex  $v_e$  for each idempotent  $e$  of the monoid;*
- *For each irreducible element  $x$  in the monoid there is an arrow from  $v_{\text{fix}(x)}$  to  $v_{\text{rfix}(x)}$ .*

Sage : generic Algo + examples...

## Conclusion

General strategy:

- Find combinatorial models for algebras and representations
- As simple as possible, but no simpler
- Concrete and effective
- Use representation theory and computer exploration as a guide
- Find the right point of view where proofs become trivial

Discussions?

- Integration of Jean-Éric's Semigroupe package into Sage
- Simple permutations and cutting poset
- Endomorphisms of the Boolean lattice