

Sorting monoids on Coxeter groups

A computer exploration with Sage-Combinat

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arXiv:0711.1561v1 [math.RT]

arXiv:0804.3781v1 [math.RT]

arXiv:0912.2212v1 [math.CO]

+ research in progress

Sage-Combinat (combinat.sagemath.org)

- 50+ research articles
- Sponsors: ANR, PEPS, NSF, Google Summer of Code
- Sage: 300 tickets / 100k lines integrated in Sage
- MuPAD: 115k lines of MuPAD, 15k lines of C++, 32k lines of tests, 600 pages of doc
- Nicolas Borie, Daniel Bump, Jason Bandlow, Adrien Boussicault, Vincent Delecroix, Paul-Olivier Dehaye, Tom Denton, Dan Drake, Teresa Gomez Diaz, Mike Hansen, Ralf Hemmecke, Florent Hivert, Brant Jones, Sébastien Labbé, Yann Laigle-Chapuy, Andrew Mathas, Gregg Musiker, Steven Pon, Franco Saliola, Anne Schilling, Mark Shimozono, Nicolas M. Thiéry, Justin Walker, Qiang Wang, Mike Zabrocki, ...

Bubble (anti) sort algorithm

1234

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1243

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1423

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4123

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4312

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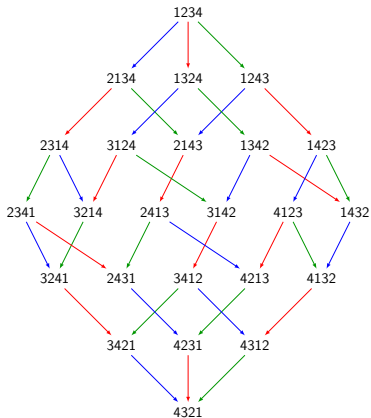
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Underlying combinatorics: right permutohedron

Bubble (anti) sort algorithm

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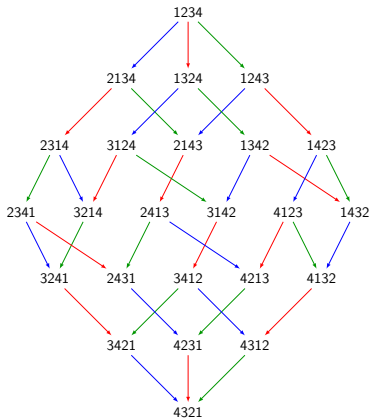
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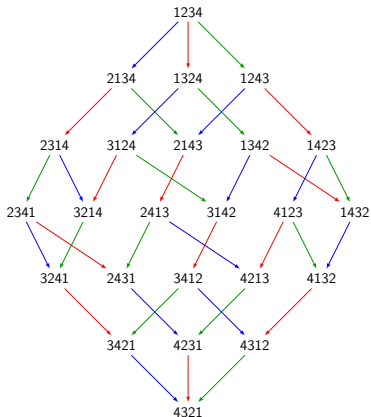


Elementary transpositions: s_1, s_2, s_3, \dots

Bubble (anti) sort algorithm

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Underlying combinatorics: right permutohedron

Elementary transpositions: s_1, s_2, s_3, \dots Relations: $s_i^2 = 1, (s_1 s_2)^3 = 1, (s_2 s_3)^3 = 1, (s_1 s_3)^2 = 1$

Coxeter groups

Definition (Coxeter group W)

Generators : s_1, s_2, \dots (simple reflections)

Relations: $s_i^2 = 1$ and $\underbrace{s_i s_j \cdots}_{m_{i,j}} = \underbrace{s_j s_i \cdots}_{m_{i,j}},$ for $i \neq j$

- Reduced word
- Length

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Orders on words and on Coxeter group elements

Definition (Orders on words)

Let $u = u_1 \cdots u_k$ and $v = v_1 \cdots v_l$:

- u **left factor** of v if $v = u_1 \cdots u_k \cdots$
- u **right factor** of v if $v = \cdots u_1 \cdots u_k$
- u **factor** of v if $v = \cdots u_1 \cdots u_k \cdots$
- u **subword** of v if $v = \cdots u_1 \cdots u_2 \cdots u_k \cdots$

Definition (Orders on Coxeter group elements)

- Right weak order
- Left weak order
- Left-right weak order
- Bruhat order

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Blocks of permutations

Definition (Block of a permutation w)

- Type A : sub-permutation matrix
- Type free: J, K such that $W_J w = w W_K$
- Example: $w := 36475812$
- Simple permutation: cf. [Albert, Atkinson 05]
- $\{\text{blocks of } w\}$: sub-lattice of the Boolean lattice

Definition (HST09: Cutting poset (W, \sqsubset))

$u \sqsubset w$ if $u = w^J$ with J block

- (almost) lattice
- Möbius function: inclusion-exclusion along minimal blocks

Hecke monoid

Definition (0-Hecke monoid $H_0(W)$ of a Coxeter group W)

Generators : $\langle \pi_1, \pi_2, \dots \rangle$ (simple reflections)

Relations: $\pi_i^2 = \pi_i$ and braid relations

Theorem

$$|H_0(W)| = |W|$$

+ lots of nice properties

Motivation: simple combinatorial model (bubble sort)
 appears in Iwahori-Hecke algebras, Schur symmetric functions,
 Schubert, Kazhdan-Lusztig polynomials, and Macdonald, (affine)
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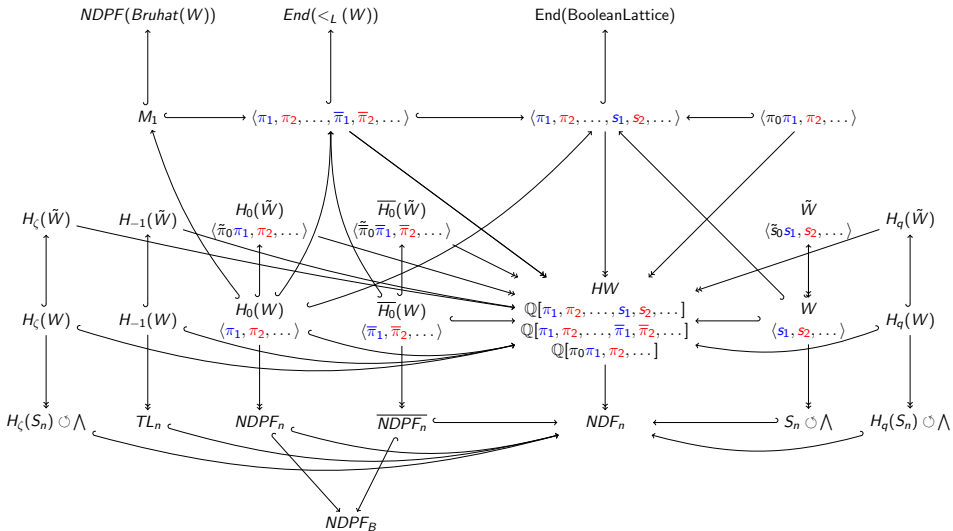
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The Big Picture



The bi-Hecke monoid

Question

$$\text{Size of } M(W) = \langle \pi_1, \pi_2, \dots, \bar{\pi}_1, \bar{\pi}_2, \dots \rangle$$

$$|M(S_n)| = 1, 3, 23, 477, 31103, ?$$

- How to attack such a problem?
- Generators and relations?
- Representation theory?

Theorem (HST08)

$M(W)$ admits $|W|$ simple / indecomposable projective modules

- Why do we care?

$$|M(W)| = \sum_{w \in W} \dim S_w \cdot \dim P_w$$

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Representation theory of algebras

Module: vector space V with a morphism $M \mapsto \text{End}(V)$

Simple module: V contains no nontrivial submodule

Indecomposable module: V cannot be written as $V = V_1 \oplus V_2$

Projective module: $V \oplus \dots = \mathbb{C}[M] \oplus \dots \oplus \mathbb{C}[M]$

Theorem (See e.g. Curtis-Reiner)

Simple modules \leftrightarrow indecomposable projective modules

Dimension formula, ...

Key role of idempotents:

- eV projective module: $V = eV \oplus (1 - e)V$
- If $f = uev$ then fM is isomorphic to a submodule of eM

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Representation theory of monoids

Definition (J -(pre)order)

$x \leq_J y$ iff $x = uyv$, for some $u, v \in M$

$x, y \in M$ are in the same J -class if $x \leq_J y$ and $y \leq_J x$

A J -class is regular iff it contains an idempotent

Theorem (See e.g. Ganyushkin, Mazorchuk, Steinberg 07)

The regular J -classes determine the simple modules.

\implies Combinatorial Representation Theory

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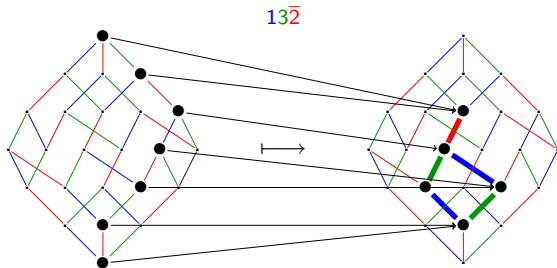
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The regular J -classes (essentially) determine the simple modules.

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Key combinatorial lemma

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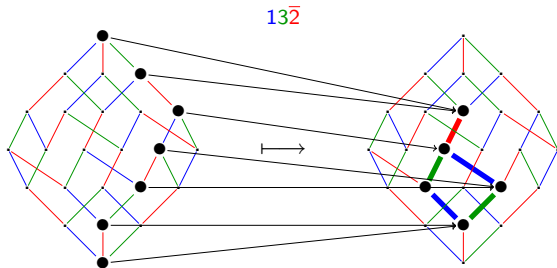
Lemma

For $f \in M(W)$ and $w \in W$: $(s_i w).f = w.f$ or $s_i(w.f)$

Proof.

Exchange property / associativity □

Key combinatorial lemma



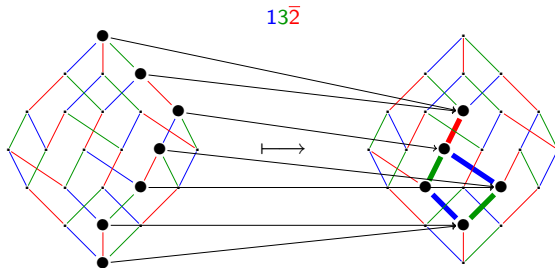
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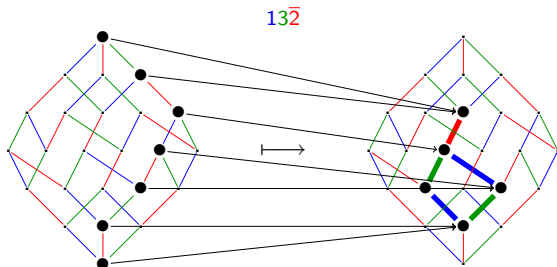
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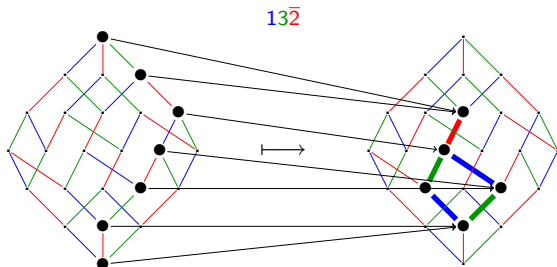
Key combinatorial lemma



Corollary

- If $w = uv$, then $(uv).f = u' (v.f)$, where $u' <_B u$
- Preservation of left order: $u \leq_L v \implies u.f \leq_L v.f$
- Preservation of Bruhat order: $u \leq_B v \implies u.f \leq_B v.f$
- f in $M(W)$ is determined by its fibers and $f(1)$

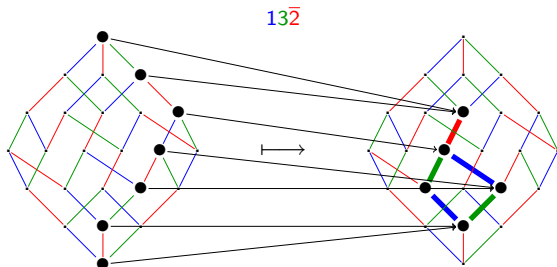
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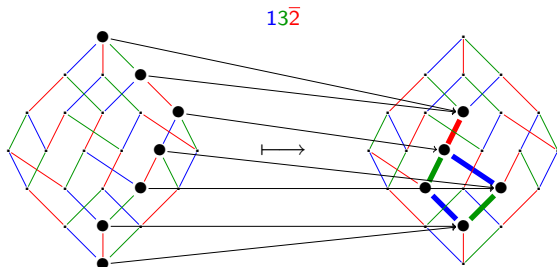
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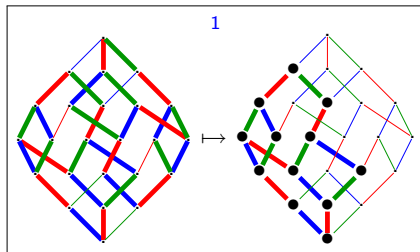
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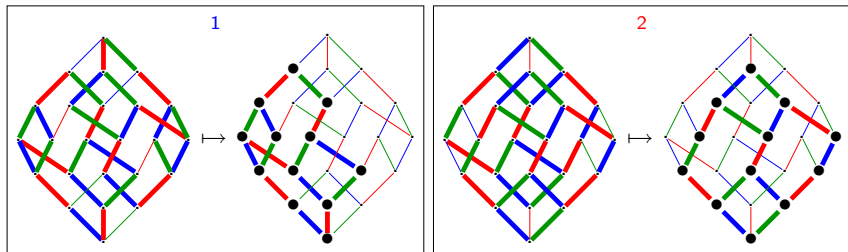
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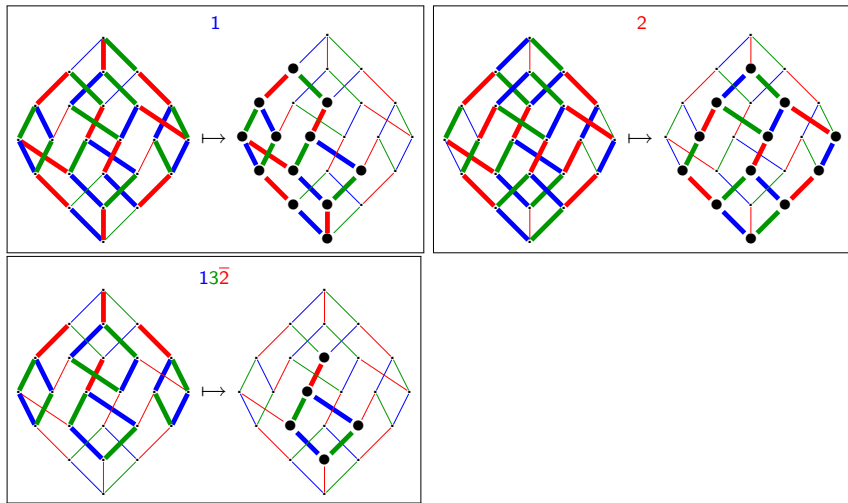
Some elements of the monoid



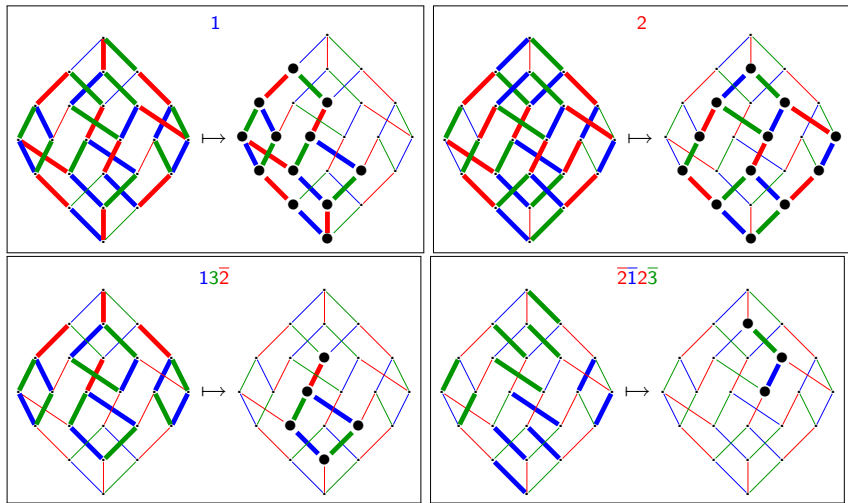
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Representation theory of $M(W)$

Theorem (HST'08)

$M(W)$ admits $|W|$ simple modules

Sketch of proof.

- M acts transitively on intervals $[u, v]_L$
- The image set of an idempotent is an interval $[u, v]_L$
- $\exists!$ e_w idempotent with image set $[1, w]_L$, for any $w \in W$
- $(e_w)_{w \in W}$: transversal of the regular J -classes
 - $f = uev$ if and only if $\text{im}(f)$ is a subinterval of $\text{im}(e)$
- The groups $e_w M e_w$ are trivial □

Problem

Dimension of simple and projective modules?

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 - $f = uev$ if and only if $\text{im}(f)$ is a subinterval of $\text{im}(e)$
- The groups $e_w M e_w$ are trivial □

Problem

Dimension of simple and projective modules?

Representation theory of $M(W)$

Theorem (HST'08)

$M(W)$ admits $|W|$ simple modules

Sketch of proof.

- M acts transitively on intervals $[u, v]_L$
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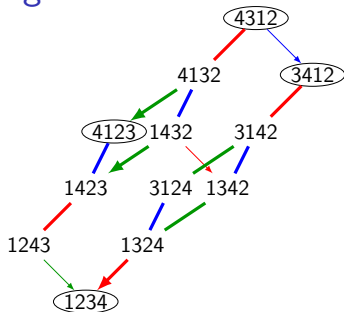
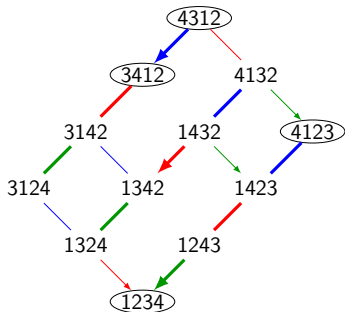
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Dimension of simple and projective modules?

Translation algebras



Definition (Translation algebra)

$$T_w := \mathbb{Q}[\pi_1, \pi_2, \dots, \bar{\pi}_1, \bar{\pi}_2, \dots] \text{ acting on } \mathbb{Q}[1, w]_R$$

- Blocks: $J = \{\}, \{1, 2\}, \{3\}, \{1, 2, 3\} \implies$ Submodules P_J
- T_w : max. algebra stabilizing all $P_J \implies$ Repr. theory
- T_w quotient of $\mathbb{Q}[M(W)]$; top: simple module S_w of M
- Dimension: inclusion-exclusion along the cutting poset
- Generating series calculation?

The “Borel” submonoid M_1

Definition

Submonoid $M_1 := \{f \in M, f(1) = 1\}$

Properties (HST'09)

- *Generated by e_w for w grassmanian*
- *Weakly increasing and contracting on Bruhat $\implies J$ -trivial*
- *Idempotents: $(e_w)_{w \in W}$*
- *$|W|$ simple modules of dimension 1*
- *Semi simple quotient: monoid algebra of (W, \vee_L)*
- *Conjugacy order among idempotents: $<_L$*
- *$\dim P_w = |\{f \in M_1, f(w) = w \dots\}|$?*

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Inducing these results to M ?

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Problem

Inducing these results to M ?

Representation theory of J -trivial monoids

Theorem (HST'09)

Combinatorial description of:

- *Simple modules*
- *Projective modules*
- *Cartan matrix*
- *Quiver*
- *q -Cartan matrix (in progress)*

in term of some statistic on M

Question

Induction from "Borel" submonoids?

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Question

Induction from “Borel” submonoids?

The path algebra of a Quiver

Definition

- Quiver: (edge labeled) graph $Q = (V, E)$
- path of length l (possibly $= 0$)

$$p := (v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \cdots \xrightarrow{e_l} v_l)$$

such that e_i is an edge from v_{i-1} to v_i .

- path algebra (category): product = concatenation if last and first vertex matches else 0.

An example

Structure theorem for finite dimensional algebras

Definition

admissible ideal: included in the ideal of path of length ≥ 2 .

Theorem

*For any (elementary) algebra A , there is a **unique quiver** Q such that A is the quotient of $\mathbb{C}Q$ by an admissible ideal I .*

Elementary algebras: simple module are all 1-dimensional.

Note: first order approximation of the algebra.

Note: the ideal I is far from being unique.

Vertices of the Quiver ?

Decomposition of the identity:

$$1 = \sum_{i \in I} f_i \quad \text{and} \quad f_i f_j = \delta_{ij} f_i;$$

Theorem (HST 09)

Construction: $(f_e \in \mathbb{C}[M])_{e \in M}$ maximal decomposition of the identity. Moreover $f_e = e + \text{smaller terms} \dots$

The vertices of the Quiver are naturally indexed by the idempotents of the monoid.

Cartan's invariants

Matrix decomposition of an algebra A :

$$x = \sum_{i,j} x_{i,j} \quad \text{where} \quad x_{i,j} = f_i x f_j.$$

$$(xy)_{i,j} = \sum_k x_{i,k} y_{k,j}$$

Definition (Cartan's Invariants)

$$C_{i,j} := \dim(f_i A f_j)$$

Automorphism sub-monoids and Cartan invariants

Automorphism sub-monoids: $\text{rAut}(x) := \{u \in M \mid xu = x\}$

Proposition

There exists a unique idempotent $\text{rfix}(x)$ such that

$$\text{rAut}(x) = \{u \in M \mid \text{rfix}(x) \leq_J u\}.$$

Same one the left ($\text{lAut}(x), \text{lfix}(x)$).

Theorem (HST09)

Cartan's invariants:

$$\dim(f_{e_1} \mathbb{C}[M] f_{e_2}) = \#\{x \in M \mid \text{lfix}(x) = e_1 \text{ and } \text{rfix}(x) = e_2\}.$$

Factorizations

Definition

Let $x \in M$ non idempotent and $e := \text{lfix}(x)$ and $f := \text{rfix}(x)$.
A factorization $x = uv$ is **compatible** if u and v are non-idempotent and

$$e = \text{lfix}(u), \quad \text{rfix}(u) = \text{lfix}(v), \quad \text{rfix}(v) = f.$$

$x \in M$ non idempotent is **irreducible** if there is no compatible factorizations $x = uv$.

The Quiver of (the algebra of) a J -trivial monoid

Theorem (HST 10)

The quiver of the algebra of M is the following:

- *There is one vertex v_e for each idempotent e of the monoid;*
- *For each irreducible element x in the monoid there is an arrow from $v_{\text{fix}(x)}$ to $v_{\text{rfix}(x)}$.*

Sage : generic Algo + examples...

Going further: Work in progress

- radical filtration = length of the paths, for some particular combinatorial J -trivial monoids
- generalization to R -trivial and aperiodic monoids
- fast implementation is Sage using Semigroupe

Conclusion

General strategy:

- Find combinatorial models for algebras and representations
- As simple as possible, but no simpler
- Concrete and effective
- Use representation theory and computer exploration as a guide
- Find the right point of view where proofs become trivial

Discussions?

- Integration of Jean-Éric's Semigroupe package into Sage
- Simple permutations and cutting poset
- Endomorphisms of the Boolean lattice

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