Sorting monoids on Coxeter groups

A computer exploration with `Sage-Combinat`

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IGM, June 8th of 2010

arXiv:0711.1561v1 [math.RT]
arXiv:0804.3781v1 [math.RT]
arXiv:0912.2212v1 [math.CO]
+ research in progress

Sage-Combinat (`combinat.sagemath.org`)

- 50+ research articles
- Sponsors: ANR, PEPS, NSF, Google Summer of Code
- Sage: 300 tickets / 100k lines integrated in Sage
- MuPAD: 115k lines of MuPAD, 15k lines of C++, 32k lines of tests, 600 pages of doc
- Nicolas Borie, Daniel Bump, Jason Bandlow, Adrien Boussicault, Vincent Delecroix, Paul-Olivier Dehaye, Tom Denton, Dan Drake, Teresa Gomez Diaz, Mike Hansen, Ralf Hemmecke, Florent Hivert, Brant Jones, Sébastien Labbé, Yann Laigle-Chapuy, Andrew Mathas, Gregg Musiker, Steven Pon, Franco Saliola, Anne Schilling, Mark Shimozono, Nicolas M. Thiéry, Justin Walker, Qiang Wang, Mike Zabrocki, ...

Bubble (anti) sort algorithm

1234

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1243

Bubble (anti) sort algorithm

1423

Bubble (anti) sort algorithm

4123

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4123

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4132

Bubble (anti) sort algorithm

4312

Bubble (anti) sort algorithm

4312

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4321

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Underlying combinatorics: right permutohedron

Bubble (anti) sort algorithm

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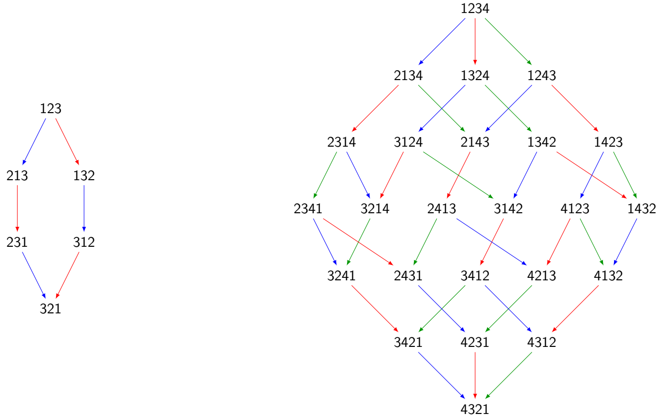
Underlying combinatorics: right permutohedron



Bubble (anti) sort algorithm

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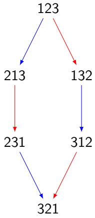
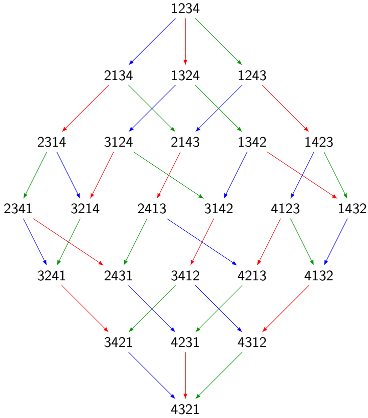
Underlying combinatorics: right permutohedron





Elementary transpositions: $s_1, s_2, s_3, \ldots$

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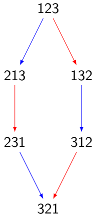

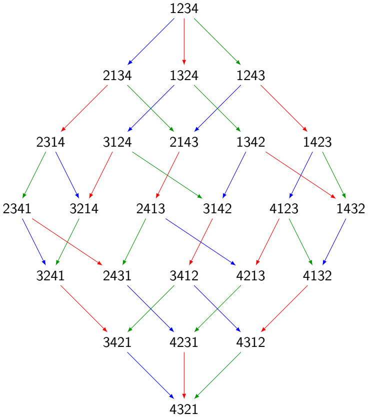

Elementary transpositions: $s_1, s_2, s_3, \ldots$

Relations: $s_i^2, (s_1 s_2)^3 = 1, (s_2 s_3)^3 = 1, (s_1 s_3)^2 = 1$

Coxeter groups

Definition (Coxeter group W)

Generators : $s_1, s_2, \ldots$ (simple reflections)

Relations: $s_i^2 = 1$ and $\underbrace{s_i s_j \cdots}_{m_{i,j}} = \underbrace{s_j s_i \cdots}_{m_{i,j}}$, for $i \neq j$

- Reduced word
- Length

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Orders on words and on Coxeter group elements

Definition (Orders on words)

Let $u = u_1 \cdots u_k$ and $v = v_1 \cdots v_l$:

- u **left factor** of v if $v = u_1 \cdots u_k \cdots$
- u **right factor** of v if $v = \cdots u_1 \cdots u_k$
- u **factor** of v if $v = \cdots u_1 \cdots u_k \cdots$
- u **subword** of v if $v = \cdots u_1 \cdots u_2 \cdots u_k \cdots$

Definition (Orders on Coxeter group elements)

- Right weak order
- Left weak order
- Left-right weak order
- Bruhat order

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Blocks of permutations

Definition (Block of a permutation w)

- Type A: sub-permutation matrix
- Type free: J, K such that $W_J w = w W_K$

- Example: $w := 36475812$
- Simple permutation: cf. [Albert, Atkinson 05] + dim 2 posets
- {blocks of w}: sub-lattice of the Boolean lattice

Definition (HST09: Cutting poset $(W, \sqsubset)$)

$u \sqsubset w$ if $u = w^J$ with J block

- (almost) lattice
- Möbius function: inclusion-exclusion along minimal blocks

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Hecke monoid

Definition (0-Hecke monoid $H_0(W)$ of a Coxeter group W)

Generators : $\langle \pi_1, \pi_2, \ldots \rangle$ (simple reflections)
Relations: $\pi_i^2 = \pi_i$ and braid relations

Theorem

$|H_0(W)| = |W|$
+ *lots of nice properties*

Motivation: simple combinatorial model (bubble sort)
appears in Iwahori-Hecke algebras, Schur symmetric functions, Schubert, Kazhdan-Lusztig polynomials, and Macdonald, (affine) Stanley symmetric functions, mathematical physics, Schur-Weyl duality for quantum groups, representations of $GL(\mathbb{F}_q)$, ...

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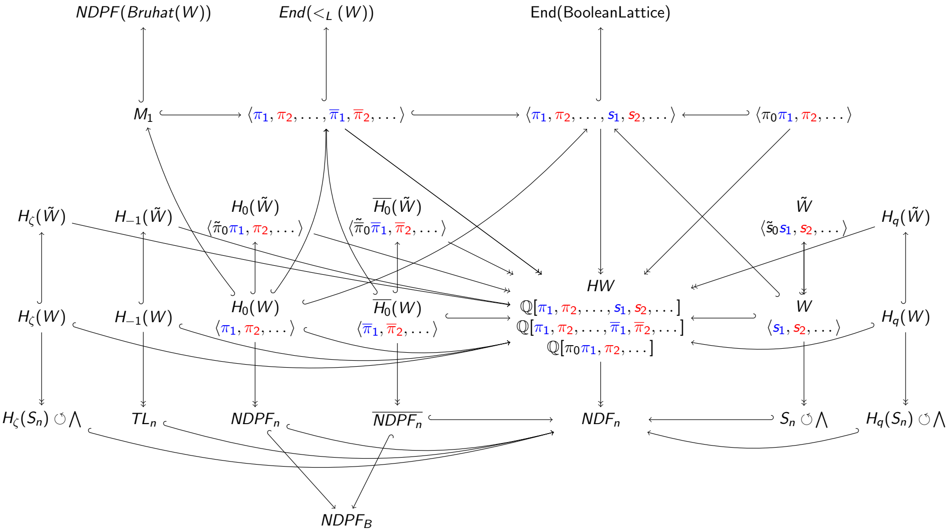
The Big Picture



The bi-Hecke monoid

Question

Size of $M(W) = \langle \pi_1, \pi_2, \dots, \overline{\pi}_1, \overline{\pi}_2, \dots \rangle$

$|M(S_n)| = 1, 3, 23, 477, 31103, ?$

- How to attack such a problem?
- Generators and relations?
- Representation theory?

Theorem (HST08)

$M(W)$ admits $|W|$ simple / indecomposable projective modules

- Why do we care?

$$|M(W)| = \sum_{w \in W} \dim S_w . \dim P_w$$

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Theorem (HST08)

$M(W)$ *admits* $|W|$ *simple / indecomposable projective modules*

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Representation theory of algebras

Module: vector space V with a morphism $M \mapsto \mathrm{End}(V)$
Simple module: V contains no nontrivial submodule
Indecomposable module: V cannot be written as $V = V_1 \oplus V_2$
Projective module: $V \oplus \cdots = \mathbb{C}[M] \oplus \cdots \oplus \mathbb{C}[M]$

Theorem (See e.g. Curtis-Reiner)

Simple modules $\leftrightarrow$ *indecomposable projective modules*
Dimension formula, ...

Key role of idempotents:

- eV projective module: $\quad V = eV \oplus (1-e)V$
- If $f = uev$ then fM is isomorphic to a submodule of eM

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Representation theory of monoids

Definition (J-(pre)order)

$x \leq_J y$ iff $x = uyv$, for some $u, v \in M$
$x, y \in M$ are in the same J-class if $x \leq_J y$ and $y \leq_J x$
A J-class is regular iff it contains an idempotent

Theorem (See e.g. Ganyushkin, Mazorchuk, Steinberg 07)

The regular J-classes determine the simple modules.

Definition (Schüztenberger)

Aperiodic monoid: no trivial subgroup

$\Longrightarrow$ Combinatorial Representation Theory

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Key combinatorial lemma

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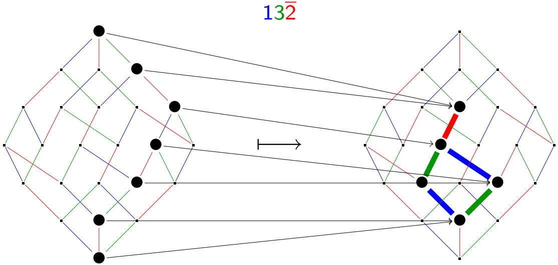


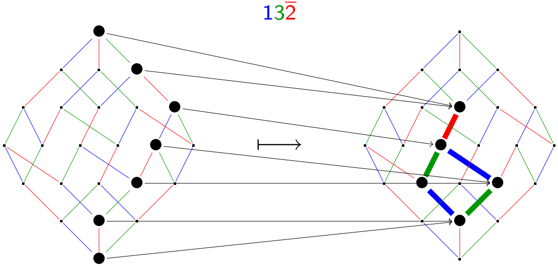
Lemma

For $f \in M(W)$ and $w \in W$: $\quad (s_i w).f = w.f \quad$ *or* $\quad s_i\,(w.f)$

Proof.

Exchange property / associativity □

Key combinatorial lemma



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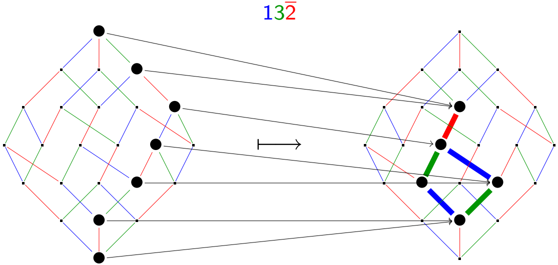

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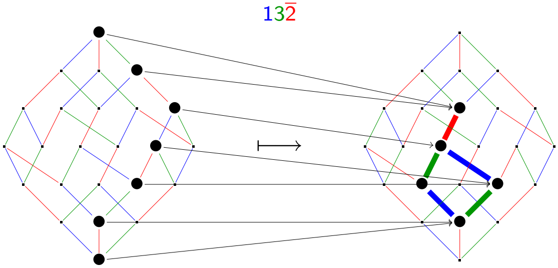

Corollary

- *If $w = uv$, then $(uv).f = u' \, (v.f)$, where $u' <_B u$*
- *Preservation of left order: $u \leq_L v \Longrightarrow u.f \leq_L v.f$*
- *Preservation of Bruhat order: $u \leq_B v \Longrightarrow u.f \leq_B v.f$*
- *f in $M(W)$ is determined by its fibers and $f(1)$*

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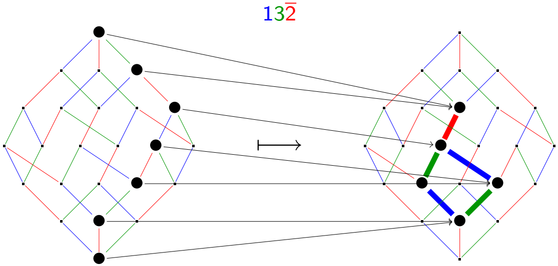

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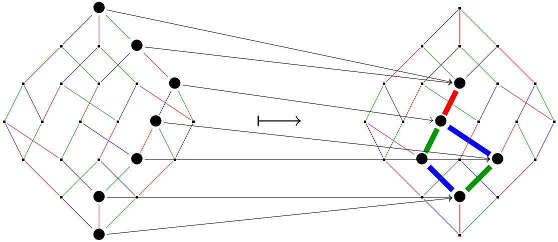

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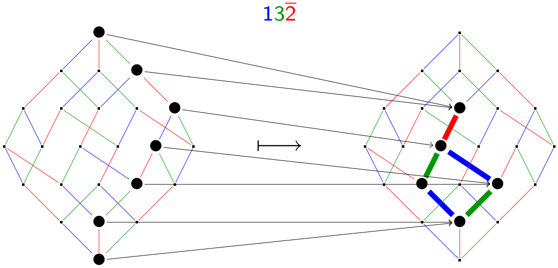

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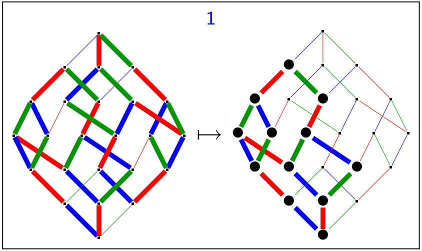

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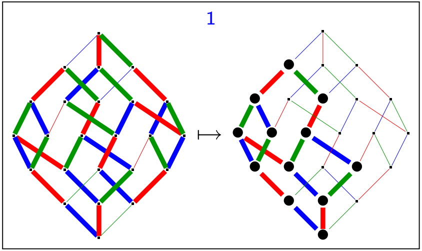

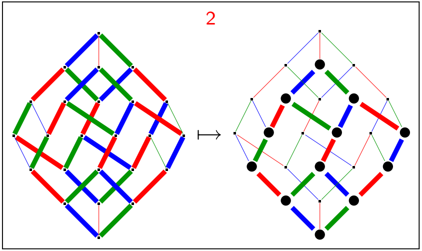

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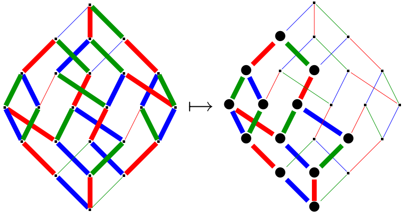

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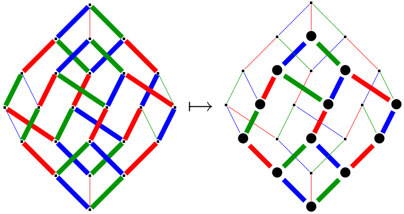
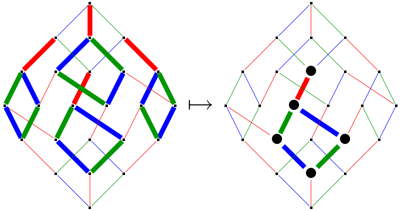


$13\overline{2}$



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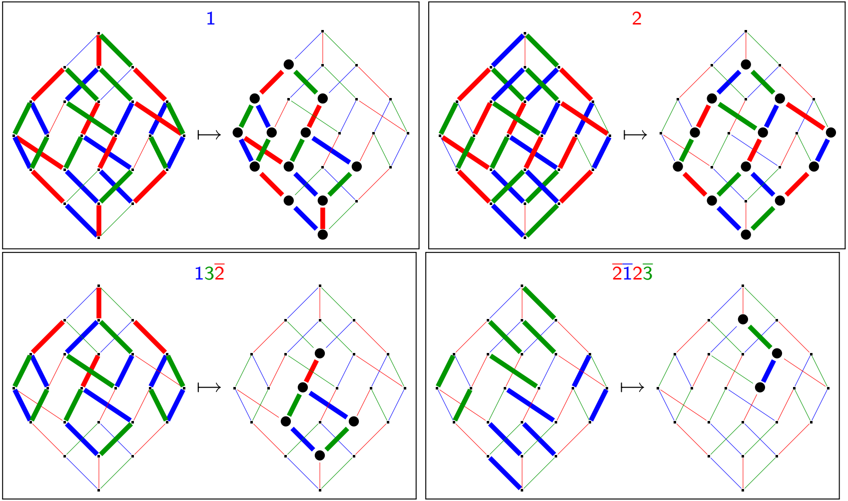

Representation theory of $M(W)$

Theorem (HST'08)

$M(W)$ admits $|W|$ simple modules

Sketch of proof.

- M acts transitively on intervals $[u, v]_L$
- The image set of an idempotent is an interval $[u, v]_L$
- $\exists!$ e_w idempotent with image set $[1, w]_L$, for any $w \in W$
- $(e_w)_{w \in W}$: transversal of the regular J-classes
  - $f = uev$ if and only if $\mathrm{im}(f)$ is a subinterval of $\mathrm{im}(e)$

□

Problem

Dimension of simple and projective modules?

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The "Borel" submonoid M_1

Definition

Submonoid $M_1 := \{f \in M, f(1) = 1\}$

Properties (HST'09)

- *Weakly increasing and contracting on Bruhat* $\Longrightarrow$ *J-trivial*
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- $|W|$ *simple modules of dimension* 1
- *Semi simple quotient: monoid algebra of* $(W, \vee_L)$
- *Conjugacy order among idempotents:* $<_L$
- $\dim P_w = |\{f \in M_1, f(w) = w \dots\}|$?

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Inducing these results to M?

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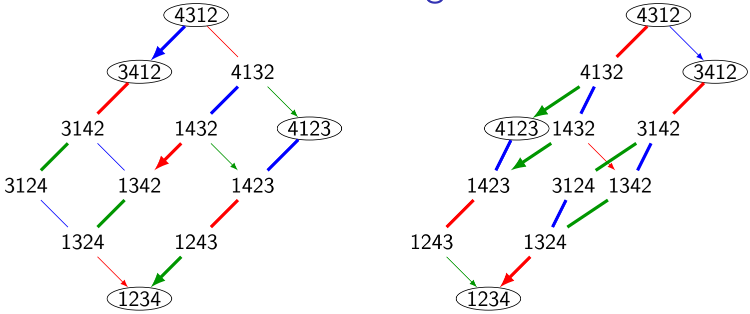
Translation algebras

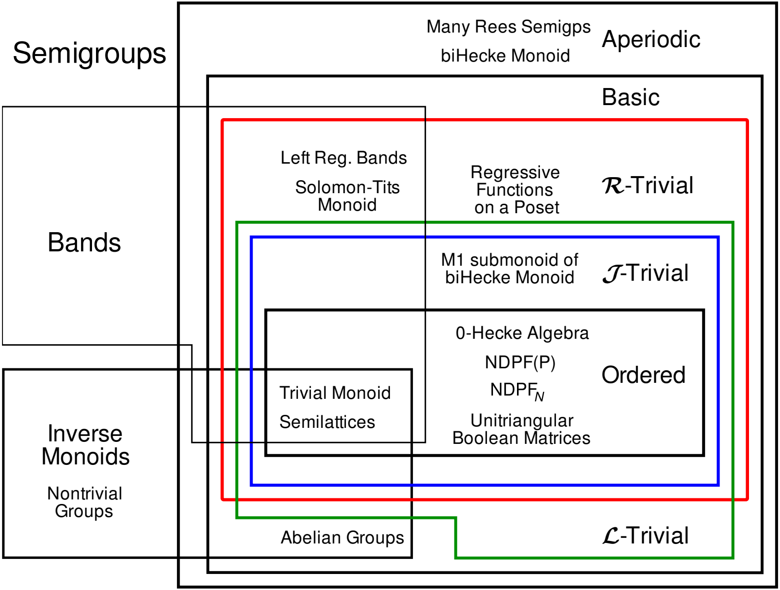


Definition (Translation algebra)

$T_w := \mathbb{Q}[\pi_1, \pi_2, \ldots, \overline{\pi}_1, \overline{\pi}_2, \ldots]$ acting on $\mathbb{Q}.[1, w]_R$

- Blocks: $J = \{\}, \{1, 2\}, \{3\}, \{1, 2, 3\}$ $\implies$ Submodules P_J
- T_w: max. algebra stabilizing all P_J $\implies$ Repr. theory
- T_w quotient of $\mathbb{Q}[M(W)]$; top: simple module S_w of M
- Dimension: inclusion-exclusion along the cutting poset
- Generating series calculation?

Classes of monoids and representation theory



Representation theory of J-trivial monoids

Theorem (HST'09)

Combinatorial description of:

- *Simple modules*
- *Projective modules*
- *Cartan matrix*
- *Quiver*
- *q-Cartan matrix (in progress)*

in term of some statistic on M

The path algebra of a Quiver

Definition

- Quiver: (edge labeled) graph $Q = (V, E)$
- path of length l (possibly $= 0$)

$$p := (v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \cdots \xrightarrow{e_l} v_l)$$

  such that e_i is an edge from v_{i-1} to v_i.
- path algebra (category): product = concatenation if last and first vertex matches else 0.

Structure theorem for finite dimensional algebras

Definition

admissible ideal: included in the ideal of path of length ≥ 2.

Theorem

For any (elementary) algebra A, there is a **unique quiver** *Q such that A is the quotient of $\mathbb{C}Q$ by an admissible ideal I.*

Elementary algebras: simple module are all 1-dimensional.

Note: first order approximation of the algebra.

Note: the ideal I is far from being unique.

Vertices of the Quiver ?

Decomposition of the identity:

$$1 = \sum_{i \in I} f_i \qquad \text{and} \qquad f_i f_j = \delta_{ij} f_i;$$

Theorem (HST 09)

Construction: $(f_e \in \mathbb{C}[M])_{e \in M}$ *maximal decomposition of the identity. Moreover* $f_e = e +$ *smaller terms...*
The vertices of the Quiver are naturally indexed by the idempotents of the monoid.

Cartan's invariants

Matrix decomposition of an algebra A:

$$x = \sum_{i,j} x_{i,j} \quad \text{where} \quad x_{i,j} = f_i x f_j \,.$$

$$(xy)_{i,j} = \sum_k x_{i,k} y_{k,j}$$

Definition (Cartan's Invariants)

$$C_{i,j} := \dim(f_i A f_j)$$

Automorphism sub-monoids and Cartan invariants

Automorphism sub-monoids: $\mathrm{rAut}(x) := \{u \in M \mid xu = x\}$

Proposition

There exists a unique idempotent $\mathrm{rfix}(x)$ *such that*

$$\mathrm{rAut}(x) = \{u \in M \mid \mathrm{rfix}(x) \leq_J u\}\,.$$

Same one the left $(\mathrm{lAut}(x), \mathrm{lfix}(x))$.

Theorem (HST09)

Cartan's invariants:

$$\dim(f_{e_1}\mathbb{C}[M]f_{e_2}) = \#\{x \in M \mid \mathrm{lfix}(x) = e_1 \text{ and } \mathrm{rfix}(x) = e_2\}\,.$$

Factorizations

Definition

Let $x \in M$ non idempotent and $e := \mathrm{lfix}(x)$ and $f := \mathrm{rfix}(x)$.
A factorization $x = uv$ is **compatible** if u and v are non-idempotent and

$$e = \mathrm{lfix}(u), \quad \mathrm{rfix}(u) = \mathrm{lfix}(v), \quad \mathrm{rfix}(v) = f\,.$$

$x \in M$ non idempotent is **irreducible** if there is no compatible factorizations $x = uv$.

The Quiver of (the algebra of) a J-trivial monoid

Theorem (HST 10)

The quiver of the algebra of M is the following:

- *There is one vertex v_e for each idempotent e of the monoid;*
- *For each irreducible element x in the monoid there is an arrow from $v_{\mathrm{lfix}(x)}$ to $v_{\mathrm{rfix}(x)}$.*

Effective algorithm: $O(n^3)$ (maybe $O(n^2)$?)

Summary

- **Bubble sort** related monoid and algebras
- Typical question: **cardinality** ?
- Approach: **representation theory** + **computer exploration**

- Leads to interesting combinatorics:
  various **partial orders** on Coxeter groups
- Combinatorial representation theory of monoids:
  how to **eliminate linear algebra**?
- Effective algorithms and combinatorial results

Work in progress

- Radical filtration = length of the paths, for some particular combinatorial J-trivial monoids
- generalization to R-trivial and aperiodic monoids (collaboration with Denton and Berg, Bergeron, Saliola)
- Fast implementation is Sage (interface with `Semigroupe`, ...)
- Simple permutations and cutting poset