# Théorie des représentations effective des monoïdes apériodiques

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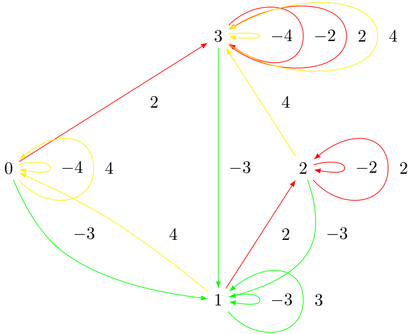

# Combinatorial Representation Theory (1)

Representation theory: lots of natural numbers !

- dimension of simple and indecomposable projective modules ($\mathfrak{S}_n, \mathrm{gl}$: Kostka numbers)
- induction and restrictions multiplicities ($\mathfrak{S}_m \times \mathfrak{S}_n \to \mathfrak{S}_{m+n}$: Littlewood-Richardson rules)
- Cartan invariant matrices and quivers ($H_n(0)$: counting permutation by descents and recoils)
- decomposition map ($H_n(q \mapsto 0)$: counting tableaux by shape and descents)

# Combinatorial Representation Theory (2)

Mostly effective: computer exploration !

Depending on

- the base field ($\mathbb{Q}$ or some extension)
- the sparsity of the multiplication table
- ...

Dimension up to 50 to 2000

# Several recent examples are monoid algebras

- 0-Hecke algebras (Norton, Carter, Krob-Thibon, Duchamp-Hivert-Thibon, Fayers, Denton)
- Non-decreasing parking function (Denton-Hivert-Schilling-T)
- Solomon-Tits algebras (Schocker, Saliola)
- Left Regular Bands (Brown) ...

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... but this fact is seldom used ...

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- Link with the representation theory
- Derive algorithm for computing the Cartan matrix

# A simple example: Order preserving functions on the chain

Definition

$f : \{1, \dots, n\} \longmapsto \{1, \dots, n\}$ is **order preserving** if:

$$i \leq j \implies f(i) \leq f(j)$$

Example

The order preserving functions on $\{1 < 2 < 3\}$:

$$\{111, 112, 113, 122, 123, 133, 222, 223, 233, 333\}$$

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## Remark

*If $f, g$ are order preserving, then so is $fg$.*
*Hence, the set $\mathcal{O}_n$ of such functions is a* **monoid** *!*

This still works if $\leq$ is replaced by a partial order

# Understanding the multiplication?

First approach: the multiplication table:

| * | 111 | 112 | 113 | 122 | 123 | 133 | 222 | 223 | 233 | 333 |
|---|---|---|---|---|---|---|---|---|---|---|
| 111 | 111 | 111 | 111 | 111 | 111 | 111 | 222 | 222 | 222 | 333 |
| 112 | 111 | 111 | 111 | 112 | 112 | 113 | 222 | 222 | 223 | 333 |
| 113 | 111 | 112 | 113 | 112 | 113 | 113 | 222 | 223 | 223 | 333 |
| 122 | 111 | 111 | 111 | 122 | 122 | 133 | 222 | 222 | 233 | 333 |
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| 133 | 111 | 122 | 133 | 122 | 133 | 133 | 222 | 233 | 233 | 333 |
| 222 | 111 | 111 | 111 | 222 | 222 | 333 | 222 | 222 | 333 | 333 |
| 223 | 111 | 112 | 113 | 222 | 223 | 333 | 222 | 223 | 333 | 333 |
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# The Cayley graph of a monoid

## Remark

*Thanks to associativity, it is sufficient to consider products*

$$xg\,, \quad \textit{for } x \in M \textit{ and } g \textit{ a generator}$$

## Definition (Cayley graph)

Graph with edges $x \stackrel{g}{\longmapsto} xg$

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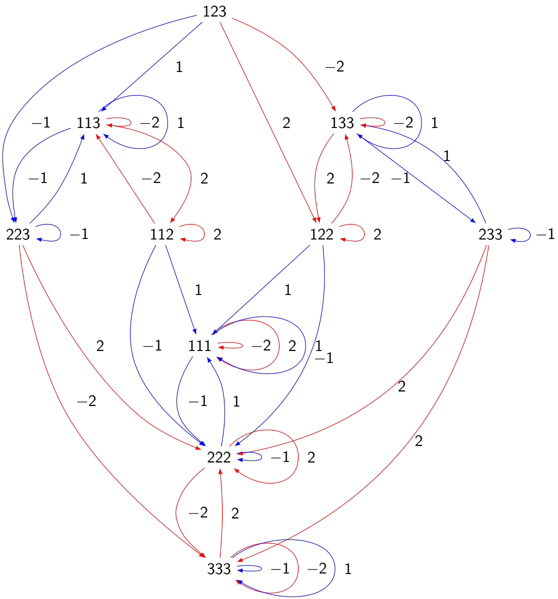
## Example

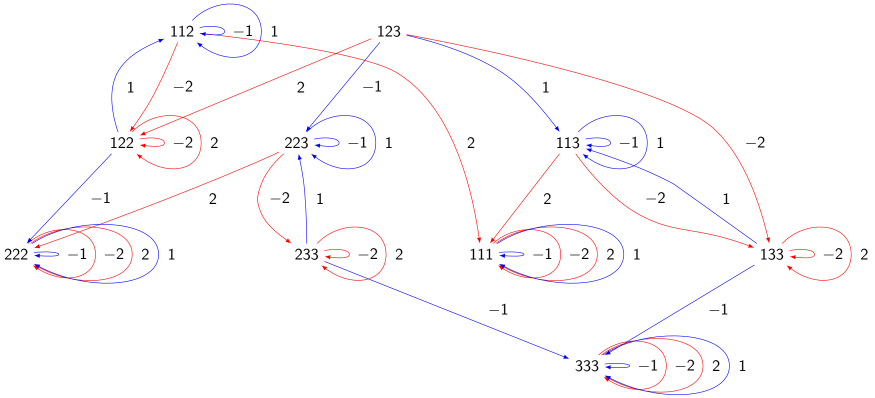
Canonical generators for $\mathcal{O}_3$:

$$\pi_1 = 113, \qquad \pi_2 = 122$$
$$\pi_{-1} = 133, \qquad \pi_{-2} = 223$$

# The right Cayley graph of $\mathcal{O}_3$



# The left Cayley graph of $\mathcal{O}_3$



# Combinatorial Module $X$ of $M$

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Finite set $X$ with an action of $M$ on $X$

Described by its Cayley graph (an automaton)
Equivalently: representation of $M$ as monoid of functions in $X^X$

## Example

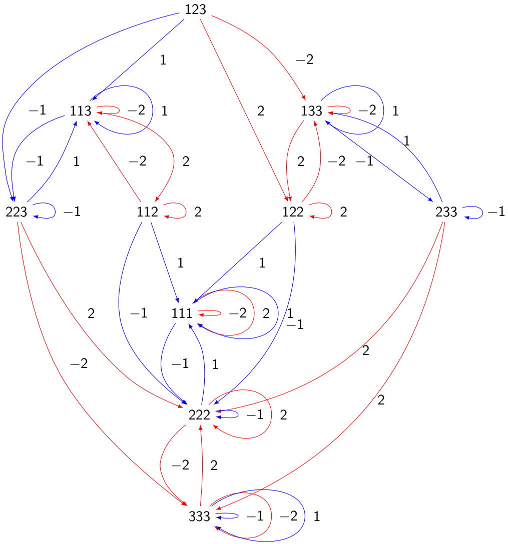
Regular representation of $M$ acting on $X = M$ (associativity!)

## Problem

*Describe all modules / representations of $M$?*

# Submodules

$X' \subset X$ is a **submodule** if it is stable under the action of $M$



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- $\mathcal{R}$-**trivial monoid**: all $\mathcal{R}$-classes are trivial

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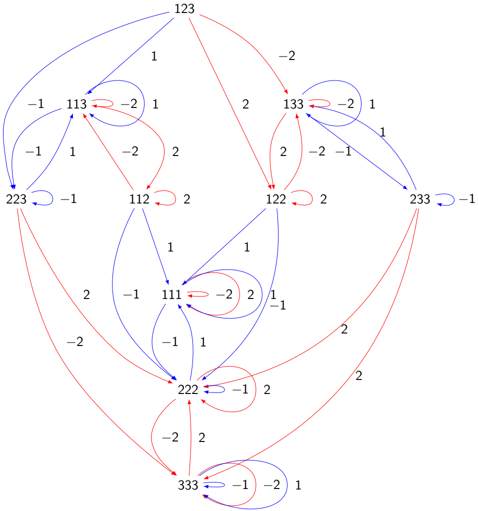
Submodule of $X$:

- Union of $\mathcal{R}$-classes of $X$
- Order ideal in $\mathcal{R}$-preorder

# Quotients by submodules

**Quotient** by submodule $X' \subset X$: $X \backslash X' \cup \{\emptyset\}$



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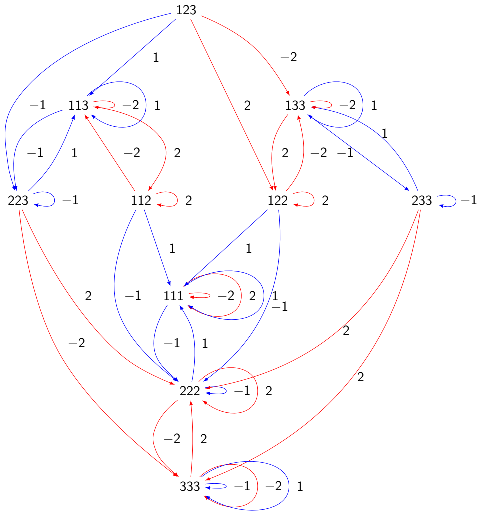


$\mathcal{R}$-classes: smallest subquotients

# Left-right Cayley graph, $\mathcal{J}$-preorder

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- *Why do we get several times the same module?*
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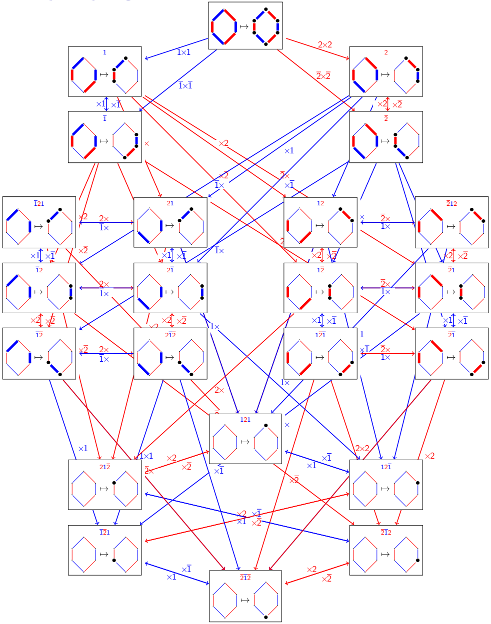
- **Left-right Cayley graph**
- $\mathcal{J}$**-class**
- $\mathcal{J}$**-preorder**
- $\mathcal{J}$**-trivial**

# Example: the left-right Cayley graph for $\mathcal{O}3$

# The left-right Cayley graph for the biHecke monoid for $\mathfrak{S}_2$



# The eggbox picture

## Proposition

*Let $J$ be a $\mathcal{J}$-class. Then,*

$$J \quad \approx_{M-\mathrm{mod}-M} \quad L \times R$$

*where $L$ and $R$ are respectively left and right classes*

*If $e$ is an idempotent:*

$$\mathcal{J}(e) = \mathcal{L}(e)\mathcal{R}(e)$$

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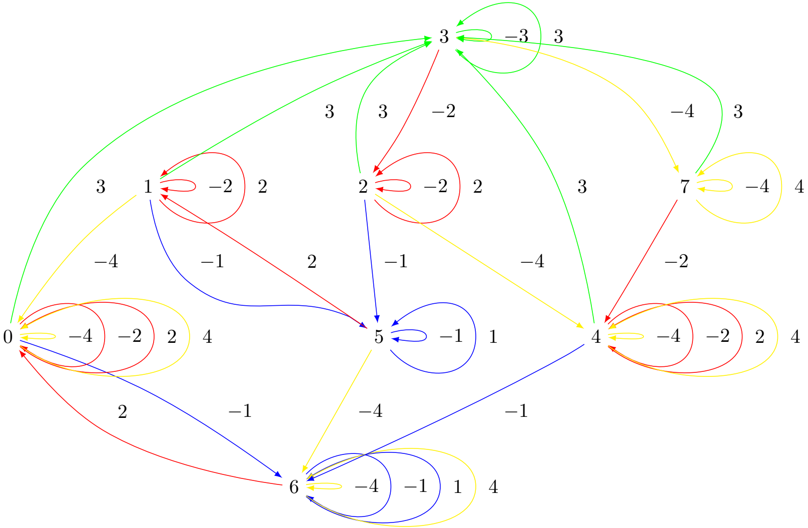
Note: unless $M$ is **aperiodic**: there are in fact groups in the boxes

# Example: a left module for the biHecke monoid for $\mathfrak{S}_5$



# Linear representations

### Definition (Module)

Vector space $V$ with an action of $M$ on $V$ by linear operators

Equivalently: linear representation of $M$ as submonoid of $M_n(K)$

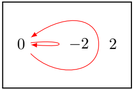
### Example

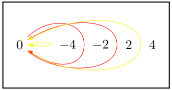
Linear module $V = \mathbb{Q}X$ associated to $X$

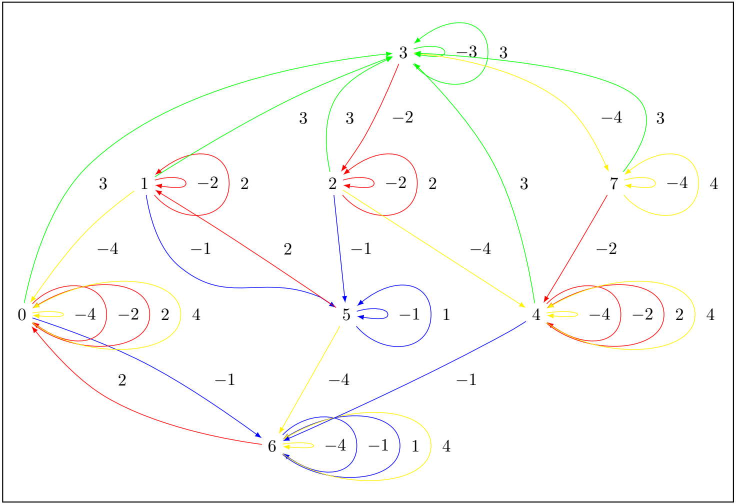
### Definition

- Submodule, simple module
- Quotient module

# Finding submodules?



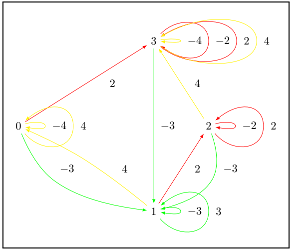



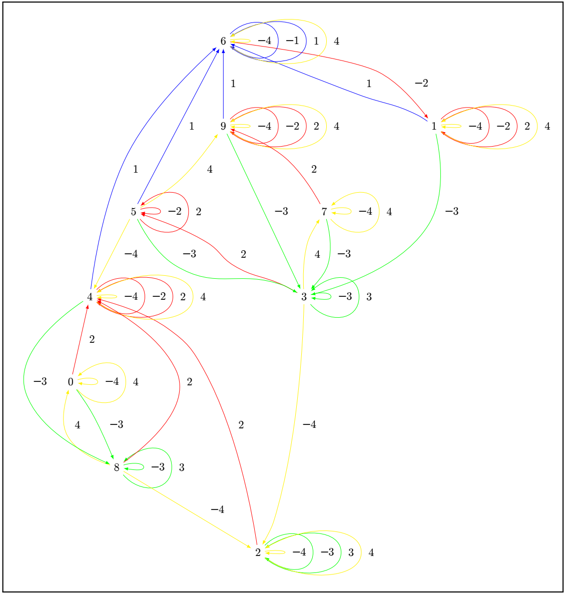
# Projective module associated to a simple module $S$

## Definition

$S$ together with everything that can appear below $S$ when embedding $S$ into another module $M$

# Embedding submodules by linear algebra



# What's known about linear representations?

## Finite groups

- Semi-simple: simple = projective
- Character theory
- Fast $o(n)$ algorithms

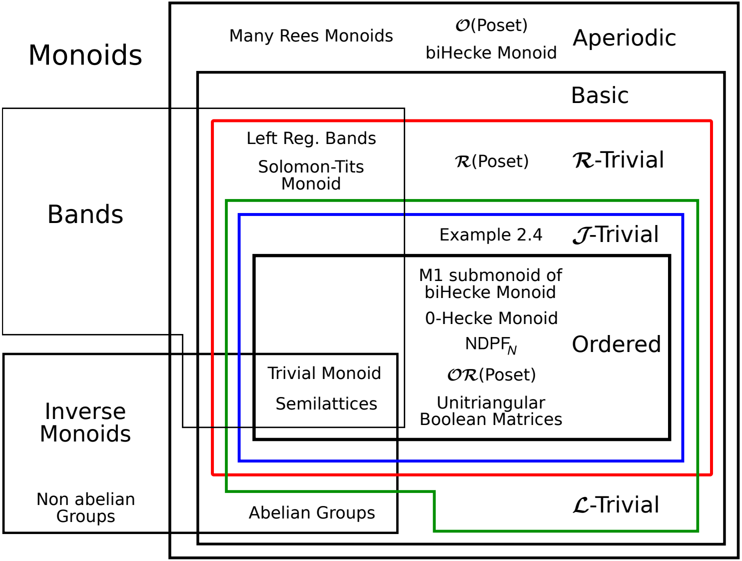
## Finite dimensional algebras

- One-to-one correspondance Simple - Projective modules
- Algorithmic: minimal polynomial, linear algebra: $O(n^3)$
- In practice: dimension $\leq 1000$

## Monoids

- In progress, thanks to Franco :-)

# Zoology of monoids



# Goal for the rest of the talk

For an aperiodic monoid, calculate

- **Cartan matrix**
- Projective modules
- Quiver
- Radical / socle filtration

# Analogue of the $\mathcal{R}$-preorder

## Definition (Composition series)

$$\{0\} = V_0 \subset \cdots \subset V_\ell = V$$

such that $V_{k+1}/V_k$ is simple

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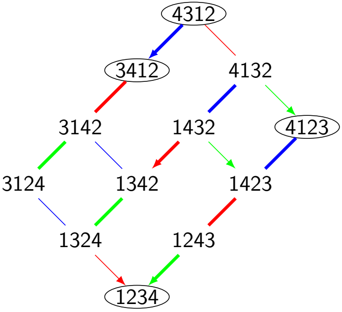


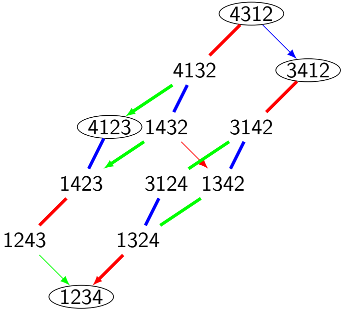

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## Proposition

*Composition series are not unique*
*The multiset $\{\{[V_{k+1}/V_k]\}\}$ of the factors is*

# Analogue of $\mathcal{J}$-preorder

## Definition

$A$: finite dimensional algebra (e.g. $A = \mathbb{Q}[M]$)
$A$ is an $A$-mod-$A$ module (or $A^{\mathrm{op}} \otimes A$-module)
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## Proposition (Analogue of the eggbox picture)

$$A_{k+1}/A_k \approx_{A-\mathrm{mod}-A} L \otimes R$$

*where $L$ is a simple left module and $R$ is a simple right module*

# Cartan matrix

## Definition

$C = (c_{i,j})_{i,j}$, with:

$$c_{i,j} \quad = \quad |\{k,\ A_{k+1}/A_k \ \approx_{A-\mod -A}\ S_i \otimes S_j\}|$$

Equivalent definitions:

- On the left: $[P_j] = \sum_i c_{i,j}[S_i]$
- On the right: $[P_i] = \sum_j c_{i,j}[S_j]$
- Dimension of sandwich by idempotents: $c_{i,j} = \dim e_i A e_j$

# Usual approach by orthogonal idempotents

1. Build a decomposition of the identity into orthogonal idempotents $e_i$
2. Compute $e_i A e_j$
3. Build the projective modules as $e_i A$

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## Problem

*Non trivial construction!*

- 0-*Hecke in type A: combinatorial formula [Denton'10]*
- $\mathcal{R}$*-trivial: recursive formula [Berg,Bergeron,Bhargava,Saliola'10]*
- *Aperiodic?*
- *Algebra: may require arbitrary algebraic extensions*

Idempotent free approach?

# Special case: $\mathcal{J}$-trivial monoids [Denton,Hivert,Schilling,T'11]

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Idem for the radical and quiver.

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Problem

*Radical filtration?*

# Cartan matrix of aperiodic monoids using the eggbox picture

## Remark

- *The composition series of $\mathbb{Q}[M]$ refines the decomposition of $M$ into $\mathcal{J}$-classes*
- *For $J$ a $\mathcal{J}$-class of the form $L \times R$:*

$$J \quad \approx_{\mathbb{Q}[M]-\mathrm{mod}-\mathbb{Q}[M]} \quad \mathbb{Q}L \otimes \mathbb{Q}R$$

## Proposition (T.)

*$M_L$: decomposition matrix of left modules into simples*
*$M_R$: decomposition matrix of right modules into simples*
*Then, $C = M_L M_R$*

Remark: $M_L$ is upper triangular; $M_R$ is lower triangular

# Algorithm

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3. Compute the composition series of each type of left and right class module:
   Find all embeddings of the simple modules; quotient out; repeat
4. Gather all the results and calculate $C = M_L M_R$

# Consequences

- No algebraic extension of $\mathbb{Q}$ needed (expected)
- Split the linear algebra in small chunks (parallelization, ...)
- Take advantage of the redundance
- Computation of the representation theory of a monoid of size 47000 in two hours

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But: not purely combinatorial

- Description?
- Quiver?
- Socle/Radical filtration?
- Interesting examples in combinatorics?
- Generalization to any finite monoid?